An Introduction to Causal Fermion Systems and its Prospects for Baryogenesis

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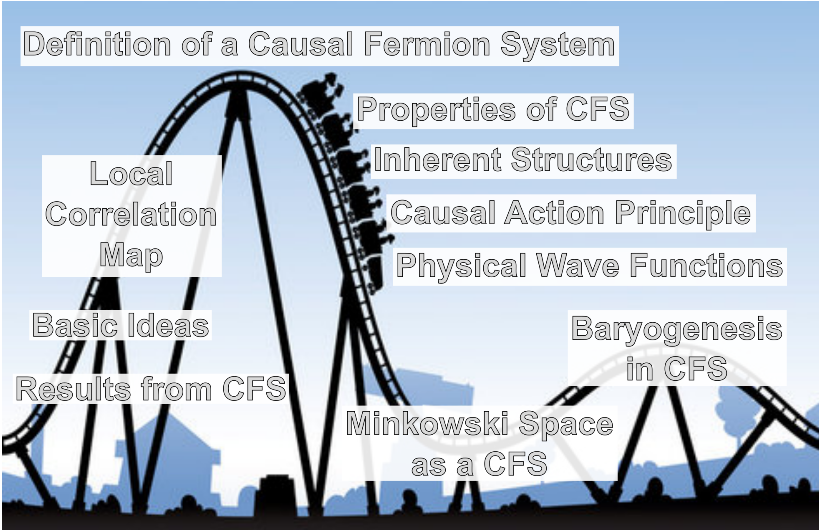

Definition of a Causal Fermion System
Properties of CFS
Inherent Structures
Local Correlation Map
Causal Action Principle
Physical Wave Functions
Basic Ideas
Baryogenesis in CFS
Results from CFS
Minkowski Space as a CFS

Goal

My goals for this talk are:

- Summarize the successes of CFS.
- Introduce the basic definitions of the theory.
- Give you an idea how to relate them to the structures we know and love with a special emphasis on the example of Minkowski space.
- Explain how the theory allows for a new mechanism for baryogenesis.

What is a causal fermion system?

- A new candidate for a unifying theory.
- Novel mathematical model of spacetime.
- Physical equations are formulated in generalized spacetimes.

Results from Theory of Causal Fermion Systems

- All physical structures are encoded in a single object.
- Standard Model gauge group and it's classical field equations in linear perturbation theory of Minkowski space. (Full dynamics of the Higgs sector remains to be worked out in detail.)
- For that to work it requires at least three generations of fermions thereby explaining one parameter of the Standard Model.
- Quantum Field Theory in non-linear perturbation theory of Minkowski space.
- Einstein equations as a third order effect
  $\Rightarrow$ Explains weakness of gravity.

Which Structures Do We Have Available In Spacetime?

- Starting point: Consider wave functions in spacetime.
  - Canonically ψ describes quantum mechanical particle (only wave character, no point particle).
  - Dynamics as described in the simplest case by Schrödinger equation (or Dirac equation, scalar wave equation, ...).
- Vector ψ in a Hilbert space $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$.
- This is not quite the right description:
  - Phase has no significance: $\psi \to e^{i\Lambda}\psi$, instead of ψ consider ray generated by ψ.
  - Local gauge invariance

$$\psi(t, \vec{x}) \to e^{i\Lambda(t,\vec{x})}\, \psi(t, \vec{x})$$

Therefore, only $|\psi(t, \vec{x})|^2$ is of physical significance; interpretation: probability density.

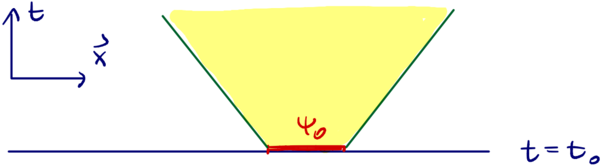
Which Structures Do We Have Available In Spacetime?

- Thus: Consider $|\psi(\mathrm{t}, \vec{\mathrm{x}})|^2$ of all wave functions as the starting point.
- General question: Suppose we know $|\psi(\mathrm{t}, \vec{\mathrm{x}})|^2$ for all the wave functions of the system, what can we say about the spacetime structures (causality, metric, fields, ...).
- Try to probe spacetime by looking at $|\psi(\mathrm{t}, \vec{\mathrm{x}})|^2$. (Here "probing" should be thought of as a mathematical operation; no collapse of the wave function involved.)

What Information Is Encoded In This Structure?

First step: Allow for preparation of the "initial state" at time t.

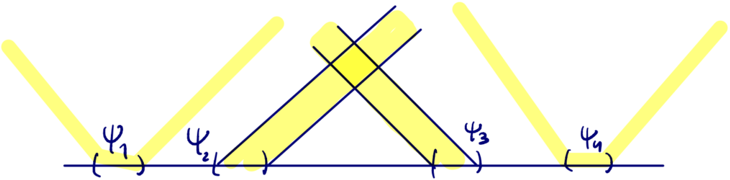
- Allows for detecting aspects of the causal structure of spacetime:



What Information Is Encoded In This Structure?

Second step: Do not allow for preparation of the "initial state".
Instead: Get by with the wave function already present.

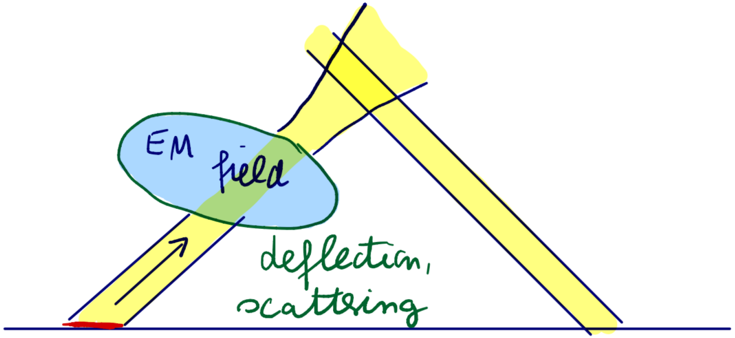
- Probing still works, provided that there are "sufficiently many" wave functions around.



- The more wave functions there are, the more information we have on spacetime (spacetime resolution).
  E.g. collection of all compactly supported solutions.
  $\rightarrow$ complete causal structure.

Which spacetime structures are fundamental?

- Allows for detecting an electromagnetic field:



- ..., ...

Formalize this idea: The local correlation operator

- Consider wave functions $\psi_1, \ldots, \psi_{\mathrm{f}} : \mathscr{M} \to \mathbb{C}$ (with $\mathrm{f} < \infty$).
- Are vectors in a Hilbert space, orthonormalize,

$$\langle \psi_{\mathrm{k}} | \psi_{\mathrm{l}} \rangle = \delta_{\mathrm{kl}} \,,$$

gives f-dim Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.

Basic object: for any point x introduce

local correlation operator $\mathrm{F}(\mathrm{x}) : \mathcal{H} \to \mathcal{H}$.

- Define matrix elements by

$$(\mathrm{F}(\mathrm{x}))^{\mathrm{j}}_{\mathrm{k}} = \overline{\psi_{\mathrm{j}}(\mathrm{x})} \psi_{\mathrm{k}}(\mathrm{x}).$$

Basis invariant:

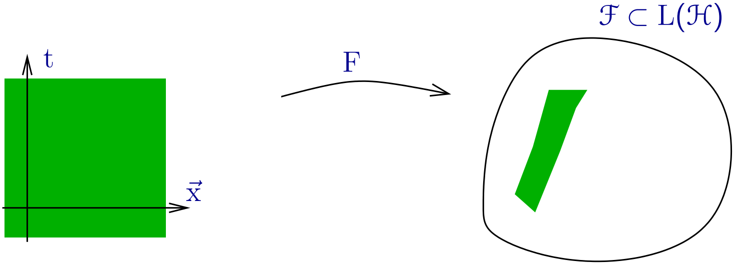
$$\langle \psi, \mathrm{F}(\mathrm{x})\, \phi \rangle_{\mathcal{H}} = \overline{\psi(\mathrm{x})} \phi(\mathrm{x}) \qquad \text{for all } \psi, \phi \in \mathcal{H}.$$

- Hermitian matrix = symmetric operator.
- Has rank at most one, is positive semi-definite.

The local correlation map

$$F(x) \in \mathcal{F} := \{F \text{ rank at most one, positive semi-definite}\}$$

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset L(\mathcal{H})$.

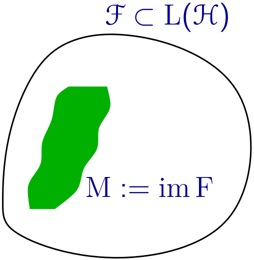


- The right side contains all the information which can be retrieved from the ensemble of wave functions.
- We consider the objects on the right as the basic physical objects.

Key Idea: Spacetime as the set of all local correlation operators

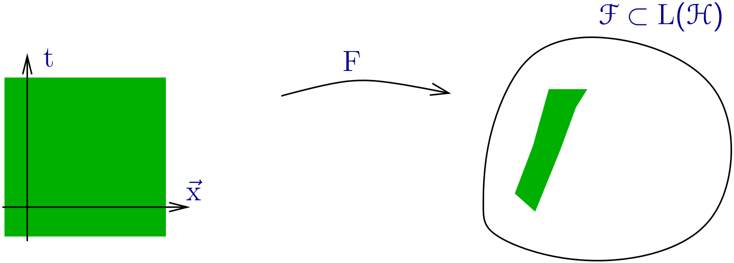
General strategy:

- Treat objects on the left as effective description (spacetime, matter fields, ...)
- Formulate a fundamental theory with the objects on the right.



A volume measure on spacetime

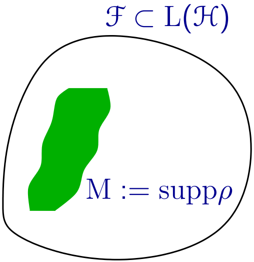
- Adding a key structure: Volume measure on spacetime.



Take push-forward measure of $\mathrm{F} : \mathscr{M} \to \mathcal{F}$,

$$\rho := \mathrm{F}_*(\mu_{\mathscr{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathscr{M}}\big(\mathrm{F}^{-1}(\Omega)\big)\text{)}$$

Spacetime



- Image of F recovered as the support of the measure,

$$\mathrm{M} := \mathrm{supp}\ \rho = \big\{ \mathrm{F} \in \mathcal{F} \mid \rho(\Omega) \neq 0 \\ \text{for every open neighborhood } \Omega \text{ of } \mathrm{x} \big\}.$$

Let's Introduce One More Spin

Let $(\mathscr{M}, \mathrm{g})$ be a Lorentzian space-time,
for simplicity 4-dimensional, globally hyperbolic,
then automatically spin:

$$(S\mathscr{M}, \prec.|.\succ) \qquad \text{spinor bundle}.$$

- $S_x\mathscr{M} \simeq \mathbb{C}^4$
- Spin scalar product

$$\prec.|.\succ_x \; : \; S_x\mathscr{M} \times S_x\mathscr{M} \to \mathbb{C}$$

is indefinite of signature (2,2).

Let's Introduce One More Spin

$$(\mathcal{D} - \mathrm{m})\psi_{\mathrm{m}} = 0 \qquad \text{Dirac equation}$$

$C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M})$ spatially compact solutions:

$$(\psi_{\mathrm{m}}|\phi_{\mathrm{m}})_{\mathrm{m}} := 2\pi \int_{\mathcal{N}} \prec\psi_{\mathrm{m}}|\not{\nu}\phi_{\mathrm{m}}\succ_x \, d\mu_{\mathcal{N}}(x) \quad \text{scalar product.}$$

Completion gives Hilbert space $(\mathcal{H}_{\mathrm{m}}, (.|.)_{\mathrm{m}})$

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\mathrm{span}(\psi_1, \ldots, \psi_{\mathrm{f}})}$$

Let's Introduce One More Spin

- To $x \in \mathbb{R}^4$ associate a local correlation operator

$$\langle \psi | F(x) \phi \rangle = - \prec \psi(x) | \phi(x) \succ_x \qquad \forall \psi, \phi \in \mathcal{H}$$

Is symmetric, rank ≤ 4, at most two positive and at most two negative eigenvalues.

- Thus $F(x) \in \mathcal{F}$ where

$$\mathcal{F} := \Big\{ F \in L(\mathcal{H}) \text{ with the properties:}$$

  - F is symmetric and has rank ≤ 4
  - F has at most 2 positive and at most 2 negative eigenvalues $\Big\}$.

Causal Fermion Systems

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space

Given parameter $n \in \mathbb{N}$ ("spin dimension")

$$\mathcal{F} := \Big\{ x \in L(\mathcal{H}) \text{ with the properties: } \begin{array}{l} \blacktriangleright\ x \text{ is symmetric and has finite rank} \\ \blacktriangleright\ x \text{ has at most } n \text{ positive} \\ \quad \text{and at most } n \text{ negative eigenvalues} \end{array} \Big\}$$

ρ a measure on $\mathcal{F}$

Comparison

Classical even dimensional tangent bundle:

- smooth 2n - dimensional manifold.
- canonical projection "assigns" an n-dim vector space to a point. Vector spaces are isomorphic but independent.
- metric defines: 1) causal structure 2) connection 3) measure supported on entire manifold.

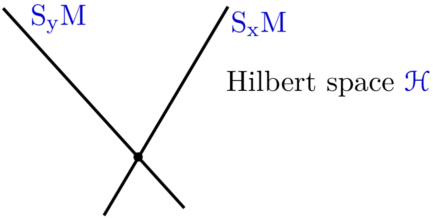
Properties of $\mathcal{F}$

- (infinite dimensional) operator manifold.
- every spacetime point comes with a 2n-dimensional vector space. These vector spaces are not independent.
- The measure is the central actor, supported on a low dimensional subset.
- Causal structure & connection independently defined on $\mathcal{F}$.

Inherent structures

- Space-time points are linear operators on $\mathcal{H}$.
- For $x, y \in M$ consider operator products xy.
  $\rightarrow$Geometric structures: causal structure, spin connection, metric connection, curvature.
- For $x \in M$, the vector space is the eigenspace of x.
  $\rightarrow$ Spinors

$$S_xM := x(\mathcal{H}) \subset \mathcal{H} \qquad \text{“spin space”}, \ \dim S_xM \leq 2n$$

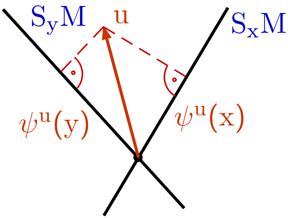


Inherent structures in spacetime

- Physical wave functions:

$$\psi^{\mathrm{u}}(\mathrm{x}) = \pi_{\mathrm{x}}\,\mathrm{u} \;\text{ with } \mathrm{u} \in \mathcal{H} \qquad \text{physical wave function}$$

$$\pi_{\mathrm{x}} : \mathcal{H} \to \mathcal{H} \qquad \text{orthogonal projection on } \mathrm{x}(\mathcal{H})$$



Causal action principle

Lagrangian $\mathcal{L}[A_{xy}] = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$

Action $\mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}[A_{xy}]\, d\rho(x)\, d\rho(y) \in [0,\infty]$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $\rho(\mathcal{F}) = \text{const}$

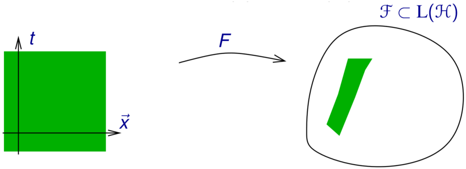
trace constraint: $\int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$

boundedness constraint: $\iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$

Local Correlation Map

We are looking for a mapping

$$\begin{aligned} F[g_{\mu\nu}, A_\mu, \dots] : \mathcal{M} &\mapsto \mathcal{F} \qquad \subset L(\mathcal{H}) \\ x &\mapsto F[g_{\mu\nu}, A_\mu, \dots](x) \end{aligned}$$

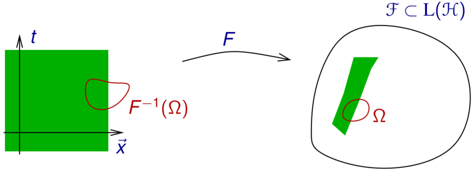


Concept:

- Left side: (approximate) effective description.
- Right side: fundamental description of the physical system.
- Map allows us to work with familiar structures on the left.

Local Correlation Map

One more thing: The space-time volume



$$\rho(\Omega) := \int_{\mathrm{F}^{-1}(\Omega)} \mathrm{d}^4\mathrm{x} = \mu\big(\mathrm{F}^{-1}(\Omega)\big)$$

- push-forward measure, is measure on $\mathcal{F}$.

Realization of the Hilbertspace in Spacetime

Apriori $\mathcal{H}$ is an abstract Hilbert space.
Physical wave functions

$$\psi(\mathrm{F}(\mathrm{x})) = \pi_{\mathrm{F}(\mathrm{x})}\,\psi \;\text{ with } \psi \in \mathcal{H}$$

where F is the local correlation map

$$\begin{aligned} \mathrm{F}[\mathrm{g}_{\mu\nu}, \mathrm{A}_\mu, \dots] : \mathcal{M} &\mapsto \mathcal{F} \qquad\quad \subset \mathrm{L}(\mathcal{H}) \\ \mathrm{x} &\mapsto \mathrm{F}[\mathrm{g}_{\mu\nu}, \mathrm{A}_\mu, \dots](\mathrm{x}) \end{aligned}$$

allow to work with suitable function spaces in $\mathcal{M}$.

How To Work With CFS (Wish)

Find a minimizing measure of the causal action.

$\Downarrow$

Determine all physical wave functions for a basis of $\mathcal{H}$.

$\Downarrow$

Find a spacetime and matter configuration that gives an approximate effective description for the causal structure of the minimizer and gives rise to equations satisfied by the physical wave functions.

How To Work With CFS (Reality)

Choose a spacetime and matter fields configuration of interest (which do not need to obey the usual physical equations).

$\Downarrow$

Select a set of functions in the spacetime at hand as a physical wave function representation of a basis of $\mathcal{H}$.

$\Downarrow$

Build the local correlation map to represent this configuration as a CFS.

$\Downarrow$

Obtain restrictions on your spacetime and matter fields from the requirement that the configuration has to be a minimizer of the causal action principle.

Dirac spinors in Minkowski space

Space-time is Minkowski space, signature $(+ - - -)$.

Space-time point $x \in \mathbb{R}^4$, need to associate operator $F(x)$.

Physical wave function representation of $\mathcal{H}$ allows us to work with functions in $\mathbb{R}^4$.

- free Dirac equation $\quad (i\gamma^k \partial_k - m)\,\psi = 0$
- probability density $\psi^\dagger \psi = \overline{\psi}\gamma^0\psi$,

  gives rise to a scalar product:

$$\langle \psi | \phi \rangle = \int_{t=\text{const}} (\overline{\psi}\gamma^0\phi)(t, \vec{x})\, d\vec{x}$$

  time independent due to current conservation

Dirac spinors in Minkowski space

- Consider a collection of one-particle wave functions

$$\mathcal{H} := <\psi_1, \ldots, \psi_{\mathrm{f}}> \qquad \text{Hilbert space}$$

$$\mathrm{b_x}(\psi, \phi) = -\overline{\psi(\mathrm{x})}\phi(\mathrm{x})$$

$$\langle \psi | \mathrm{F}(\mathrm{x}) \phi \rangle = -\overline{\psi(\mathrm{x})}\phi(\mathrm{x}) \qquad \forall \psi, \phi \in \mathcal{H}$$

local correlation operator, is self-adjoint operator in $\mathrm{L}(\mathcal{H})$

- Thus $\mathrm{F}(\mathrm{x}) \in \mathcal{F}$ where

$\mathcal{F} := \Big\{ \mathrm{F}(\mathrm{x}) \in \mathrm{L}(\mathcal{H})$ with the properties:

▷ $\mathrm{F}(\mathrm{x})$ is self-adjoint and has rank 4

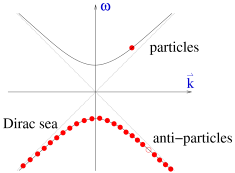
▷ $\mathrm{F}(\mathrm{x})$ has 2 positive

and 2 negative eigenvalues $\Big\}$

The causal action principle in the continuum limit

Specify vacuum:

- Choose $\mathcal{H}$ as the space of all negative-energy solutions, hence "Dirac sea".



- Introduce regularization:

  Fixes length scale ε.

Sakharov's Criteria

- Baryon number B violation.
- C-symmetry and CP-symmetry violation.
- Interactions out of thermal equilibrium.

Sakharov's Criteria

- The second criteria is trivially satisfied as the fundamental description treats particle and anti-particle states differently (while preserving the symmetries in the effective description in the continuum limit).
- In our paper (arXiv: 2111.05556) we showed that the first criterion can indeed be satisfied in the CFS setup in the form of a fermion number violation.
- It is not entirely clear how the third criteria is satisfied in the context of our new mechanism. Note however that we require deviations from FLRW for the effect to exist. This can be interpreted as a form of non-thermality.

The Mechanism - The Ingredients

- Finite regularization ε with associated regularizing vector field.
- Deviation from Dirac dynamics on cosmological scales implied by the rigid regularization.

The Mechanism - The Effect

- Dynamics in the regularization lead to a change in the "sea-level" of the Dirac sea (technically we prove a spectral flow across the zero energy level).
- Effect is identical across all three generations, across all charged fermionic sectors (i.e. the matter created is neutral w.r.t. all charges).
- Modified measures are required (instead of integrating against $\sqrt{|\mathrm{g}|}$). The dynamics of the measure leads to a non-conservation of the matter stress energy.

The Timeline

- Small ε vacuum state
- Baryogenesis
- Reheating (decay of the third generation fermions)
- FLRW evolution
- Large ε vacuum state

Novel Phenomenology

- If DM is Fermionic and weakly coupled it could survive reheating in the Fermi ground state (Fermionic Condensate Dark Matter).
- Different dynamics during the matter dominated era of the universe due to the degeneracy pressure in FCDM.
- Dynamics become dust like after structure formation when the gravitational potential wells get deep enough and the fermi gas localizes around galaxies and clusters.
- FCDM can solve the DM cusp problem in galaxies.

Summary

Spacetime and matter described by a single object:

a measure ρ on $\mathcal{F}$

- space-time $M := \text{supp}\,\rho$.
- causal relations given by spectrum of $x \cdot y$ for $x, y \in M$.

Dynamics described by causal action principle:

- minimize $\mathcal{S}$ by varying ρ.
- implicitly varies space-time and all structures therein.

applications:

- describes fundamental forces of nature.
- approach for unification of gravitation and the standard model.
- Potentially gives rise to new mechanism for fermiogenesis.

Thank You

Thank you for your attention.

For a more in depth introduction see the website
causal-fermion-system.com