PSI Summer School on Particle Physics

Bound States

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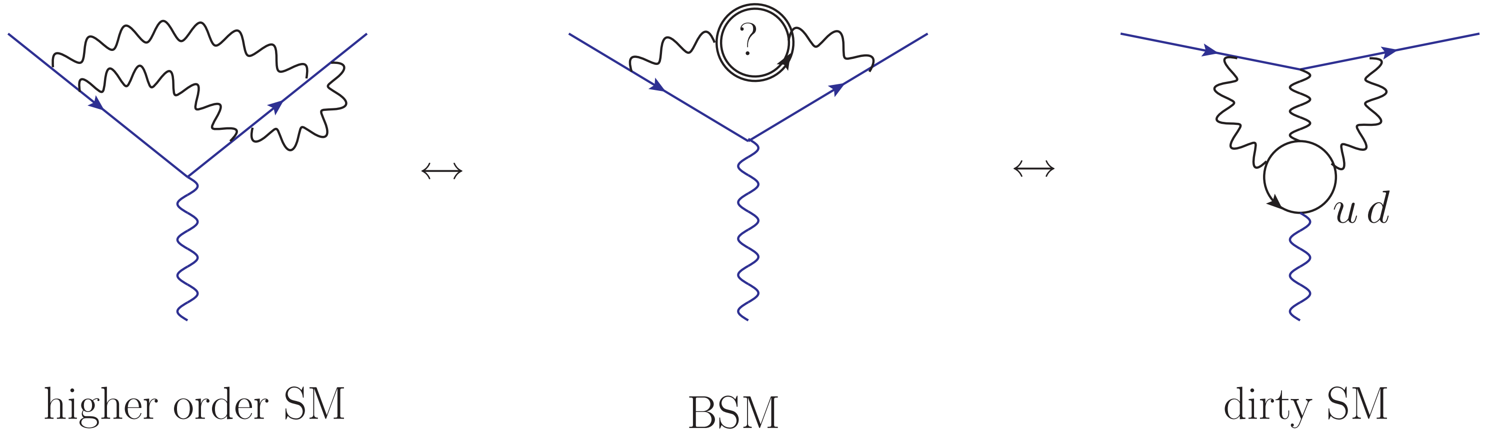
AUGUST 2012, ZUOZ

closing in on the SM $\overset{??}{\leftrightarrow}$ bound states

- usually we are testing the SM (or look for BSM) at as high energies as possible $\rightarrow$ ideally direct production of new particles
- alternatively consider virtual effects, potentially sensitive to much higher energies
- this requires the “right” observable: precise measurements and precise theory
- prime example: $(g-2)$

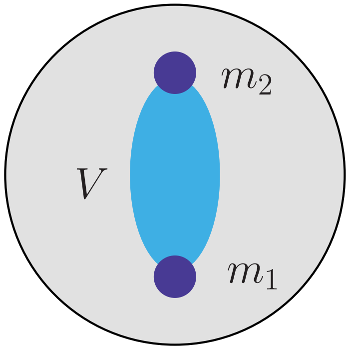


- in such tests we are looking for small effects !

- this lecture: the theory of (weakly) bound states
- motivation
  - better understanding of QFT
  - exploit potential of precise measurements to constrain/find BSM
- outlook Part I: theory (mainly Tue)
  - consider non-relativistic limit of QFT
  - explain fundamental principles of effective-theory approach
  - focus on SM part (BSM part is usually the easy bit)
  - health warning: some slides are rather technical
- outlook Part II: applications (mainly Fri)
  - heavy quark pair production near threshold
  - m_Q from $Q\bar{Q}$
  - decay ratios and HFS of $Q\bar{Q}$
  - hydrogen vs. muonic hydrogen

possible systems include:

positronium	$e^+ e^-$	$m_1 = m_2$	standard
muonium	$\mu^\pm e^\mp$	$m_1 \gg m_2$	standard
charmonium ($J/\psi, \eta_c$)	$c\,\bar{c}$	$m_1 = m_2$	$\sim$ standard
bottomonium (Υ, η_b)	$b\,\bar{b}$	$m_1 = m_2$	Υ standard, η_b only just
B_c meson	$b\,\bar{c}$	$m_1 \gg m_2$	scalar since 1998
hydrogen	$p\,e^-$	$m_1 \gg m_2$	standard
muonic hydrogen	$p\,\mu^-$	$m_1 \gg m_2$	standard
hydrogen-like	$N\,e^-$	$m_1 \gg m_2$	standard
antihydrogen	$\bar{p}\,e^+$	$m_1 \gg m_2$	since $\sim$ 1995
true muonium	$\mu^+ \mu^-$	$m_1 = m_2$	not (yet) produced
tauonium	$\tau^\pm e^\mp$	$m_1 \gg m_2$	not (yet) produced
true tauonium	$\tau^+ \tau^-$	$m_1 = m_2$	not (yet) produced
top	$t\,\bar{t}$	$m_1 = m_2$	never but nearly



- two point masses m_1 and m_2
- reduced mass $m \equiv m_1 m_2/(m_1+m_2)$
- interacting through potential $V(r) = -Z\,\alpha/r$

Schrödinger eq: $\left(-\frac{\Delta}{2m} - \frac{Z\,\alpha}{r}\right)|n\rangle = E_n|n\rangle$

Coulomb Green function: $\left(-\frac{\Delta}{2m} - \frac{Z\,\alpha}{r} - E\right) G_c(\vec{r},\vec{r}\,',E) = \delta^{(3)}(\vec{r}-\vec{r}\,')$

$G_c(\vec{r},\vec{r}\,',E)$ has poles for certain values on $E = E_n = -\frac{(Z\alpha)^2 m}{2\,n^2}$ $\Longrightarrow$ bound states

spectral representation: $$G_c(\vec{r},\vec{r}\,',E) = \underbrace{\sum_{n=1}^{\infty} \frac{\psi_n(r)\,\psi_n^*(r')}{E_n - E}}_{\text{bound states}} + \int \frac{d\vec{k}}{(2\pi)^3}\,\frac{\psi_k(r)\,\psi_k^*(r')}{k^2/m - E}$$

$G_c(\vec{r},\vec{r}\,',E)$ and $\psi_n(r) \equiv |n\rangle$ can be written in terms of Laguerre polynomials L^{2l+1}_{n-l-1}

$$\psi_{nlm}(r) \equiv |n\rangle_{lm} = \sqrt{\frac{\rho^3\,\Gamma(n-l)}{2n\Gamma(n+l+1)}}\,L^{2l+1}_{n-l-1}(\rho\,r)\,e^{-\rho\,r/2}\,(\rho\,r)^l\,Y^m_l(\theta,\phi)$$

with $\rho \equiv \dfrac{2Z\alpha m}{n} = \dfrac{2}{a_0\,n}$

scales of the problem

$\langle n|\dfrac{Z\alpha}{r}|n\rangle = \dfrac{m(Z\alpha)^2}{n^2} = \dfrac{Z\alpha}{n^2 a_0}$ Bohr radius

$\langle n|\dfrac{p}{m}|n\rangle = \langle n|v|n\rangle = \dfrac{(Z\alpha)}{n^2}$ note: $v \ll 1$ for $Z \ll \alpha \implies$ non-relativistic system !

$\langle n|\dfrac{p^2}{m^2}|n\rangle = \langle n|v^2|n\rangle = \dfrac{(Z\alpha)^2}{n^2}$ note: $\langle n|Z\alpha/r|n\rangle \not\ll \langle n|p^2/m|n\rangle$

$\langle n|\dfrac{p^2}{2m}|n\rangle = \dfrac{m(Z\alpha)^2}{2n^2} \overset{!?}{=} -E_n$ scaling $m \gg p \sim mv \gg E \sim mv^2$

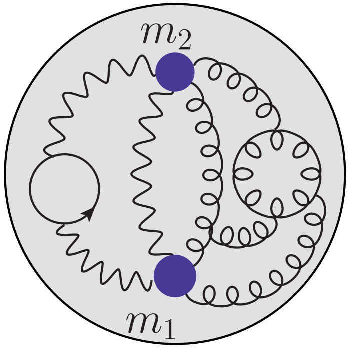
- our implicit assumption that the system is non-relativistic is justified for $(Z\alpha) \ll 1$
- there is a hierarchy of scales:

hard scale:	$m \sim 1$
soft scale:	$p \sim v \sim (Z\alpha) \ll 1$
ultrasoft scale:	$E = p^2/(2m) \sim v^2 \ll v$

- we must not treat $V(r) = -Z\alpha/r$ as perturbation, even though $(Z\alpha) \ll 1$

  starting with free Schrödinger equation and treating $-Z\alpha/r$ as perturbation will never describe a bound state
- how to go on from here:
  - recall: we will be looking at high precision!
  - either: add further effects (fine structure, hyperfine structure, recoil effects, vacuum polarization . . .) to the potential (“bottom up”, not here)
  - or: ask where does the potential come from and how is this connected to a quantum field theory (“top down”, our approach here)

$\Longrightarrow$ forget everything you know about Quantum Mechanics (for a while)



- two point masses m_1 and m_2
- reduced mass $m \equiv m_1 m_2/(m_1 + m_2)$
- interacting through Lagrangian $\mathcal{L}_{\rm QED}$ and/or $\mathcal{L}_{\rm QCD}$

- a closed solution of this problem is of course hopeless
- even if we could solve this, it would not answer all questions, since e.g. proton is not a point mass.
- goal for for the moment:
  - ignore these finite size effects
  - ignore non-perturbative effects (QCD)
  - exploit hierarchy of scales $v \ll 1$ and $(Z\alpha) \ll 1$ to make QFT tractable

Part I

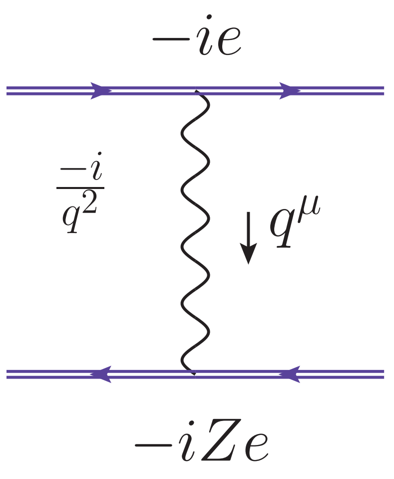
Theory

After a few slides, in a first step we will end up with

$$
\begin{aligned}
\mathcal{L}_{\text{QED}} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\Psi}(i \not{D} - m)\Psi \\
&\Downarrow \\
\mathcal{L}_{\text{NRQED}} &= \psi^\dagger \left(iD^0 + \frac{\vec{D}^2}{2m} \right) \psi + \frac{1}{8m^3} \psi^\dagger \vec{D}^4 \psi - \frac{g\, c_F}{2m} \psi^\dagger \vec{\sigma} \cdot \vec{B}\, \psi \\
&+ \frac{g\, c_D}{8m^2} \psi^\dagger \left[\vec{D} \cdot \vec{E} - \vec{E} \cdot \vec{D} \right] \psi + \frac{ig\, c_S}{8m^2} \psi^\dagger \vec{\sigma} \cdot \left[\vec{D} \times \vec{E} - \vec{E} \times \vec{D} \right] \psi \\
&+ (\psi \leftrightarrow \chi) + \mathcal{L}_{\text{light}} \\
&+ \frac{\alpha_s\, d_{ss}}{m^2} \psi^\dagger \psi\, \chi^\dagger \chi + \frac{\alpha_s\, d_{sv}}{m^2} \psi^\dagger \sigma^i \psi\, \chi^\dagger \sigma^i \chi \\
&+ \ldots \quad \text{calculable}
\end{aligned}
$$

- note: this is a strict QFT approach, in prinicple possible to include loops to any order
- $\mathcal{L}_{\text{NRQCD}}$ is an expansion of $\mathcal{L}_{\text{QED}}$ in v
- $\mathcal{L}_{\text{NRQCD}}$ gives as good a description of bound states as $\mathcal{L}_{\text{QED}}$ but is much more convenient

naive first step



exchange of photon im momentum space:

$$i\,\widetilde{V}(q) \sim \frac{(-ie)(-iZe)(-i)}{q_0^2 - \vec{q}^{\,2}} \to \frac{-iZe^2}{\vec{q}^{\,2}} + \mathcal{O}(q_0^2/q^2)$$

after Fourier transform:

$$V(r) \sim \frac{-Ze^2}{4\pi r} = -\frac{Z\alpha}{r}$$

- what happened to spinors of fermions ?
- what happened to γ^μ of vertices and $g^{\mu\nu}$ of propagator?
- let's do this properly
  - could do a Foldy-Wouthuysen transformation
  - here we will use "matching", a general technique useful in many effective theories: fix the coefficients c_j of the Lagrangian of the effective theory s.t. $\mathcal{L}_{\rm ET}$ and $\mathcal{L}_{\rm QED}$ give the same answer (up to a certain order in perturbation theory)

what is an effective theory?

theory: not a model; a framework for systematically improvable predictions

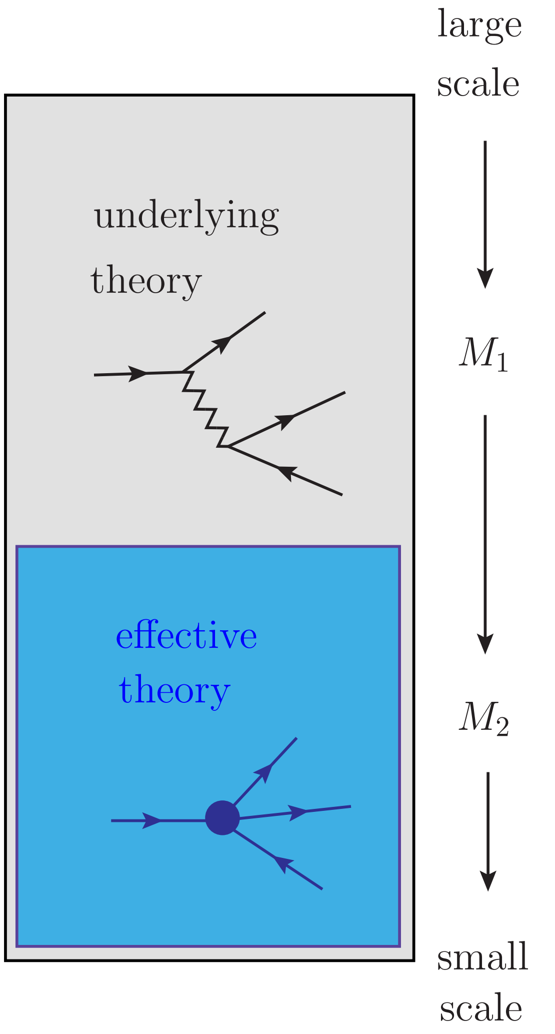
effective: not the full story; applicable only in certain circumstances $\Rightarrow$ factorization

underlying theory (UT)

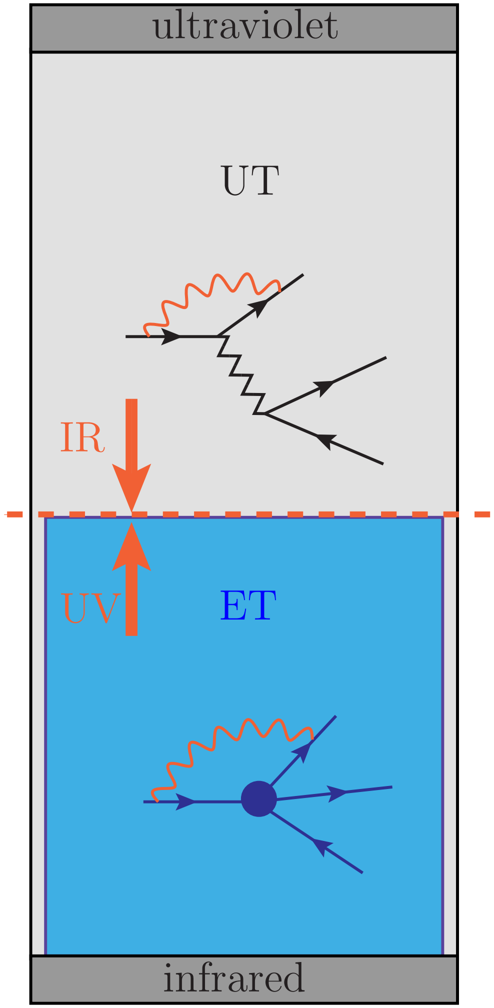
- contains dynamical (directly observable) d.o.f. of large/hard scale M_1 and small/soft scale M_2
- Lagrangian: $\mathcal{L}_{\rm UT} = \sum_i O_i(\phi_1, \phi_2)$
- observables: $f(\alpha, M_1, M_2) = \sum_n \alpha^n \, f^{(n)}_{\rm UT}(M_1, M_2)$

effective theory (ET)

- contains dynamical d.o.f. of soft scale M_2; ϕ_1 integrated out assuming $M_2/M_1 \ll 1$
- Lagrangian: $\mathcal{L}_{\rm ET} = \sum_j c_j \, O_j(\phi_2)$
- observables: $f = \sum_n \alpha^n \sum_m (M_2/M_1)^m \, f^{(n,m)}_{\rm ET}$



main features of effective theories



UV singularities $\rightarrow$ renormalize   for UT: $[O_i] \leq 4$

$$\mathcal{L}_{\rm UT} \simeq -\frac{1}{4} W_i^{\mu\nu} W^i_{\mu\nu} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$+ \sum g_w \, (\bar\psi \, \gamma^\mu \{\gamma_5\} \tau^i \, \psi) \, W^i_\mu + e \, (\bar\psi \, \gamma^\mu \, \psi) \, A_\mu + \ldots$$

integrating out the W mode
$\Rightarrow$ additional singularities at the boundary!
IR singularity of $\mathcal{L}_{\rm UT}$ = UV singularity of $\mathcal{L}_{\rm ET}$

$$\mathcal{L}_{\rm ET} \simeq -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + e \, (\bar\psi \, \gamma^\mu \, \psi) \, A_\mu$$

$$+ \sum c(M_W) \, (\bar\psi \, \{\gamma^\mu \gamma_5 \tau^i T^a\} \, \psi) \, (\bar\psi \, \{\gamma^\mu \gamma_5 \tau^i T^a\} \, \psi)$$

IR singularities $\rightarrow$ form physical observables main features of effective theories

- ever higher dimensional operators $O_j(\phi_2)$ with suppressed coefficients $c_j \sim 1/M_1^{d-4}$
- IR singularity of UT: $-\frac{1}{\epsilon}\left(\frac{M_1}{\mu}\right)^{-\epsilon} = -\frac{1}{\epsilon} + \log\frac{M_1}{\mu}$

  UV singularity of ET: $\frac{1}{\epsilon}\left(\frac{M_2}{\mu}\right)^{-\epsilon} = \frac{1}{\epsilon} - \log\frac{M_2}{\mu}$
- singularities cancel and can be predicted $\to$ logs can also be predicted $\to$
- resummation of $L \equiv \log(M_1/M_2) \gg 1$:
  - presence of terms $\alpha^n L^{2\,n}$ or $\alpha^n L^n$ invalidates expansion in α alone
  - reorganize perturbation theory:

    from a pure expansion in α (LO $\to$ NLO $\to$ NNLO ...)
    to resummed expansion, counting $\alpha L \simeq 1 \not\ll 1$ (LL $\to$ NLL $\to$ NNLL ...)
- can have a tower of ETs, i.e. for $M_1 \gg M_2 \gg M_3 \ldots$: UT $\to$ ET `I` $\to$ ET `II` ...
- in (NR)QED: we will not integrate out whole particles ($\sim$ easy), but integrate out modes (part of a quantum field with a particular scaling)
- in (NR)QED: $M_1 \sim m$ and $M_2 \sim mv$ and $M_3 \sim mv^2$

- external particles in the bound-state system

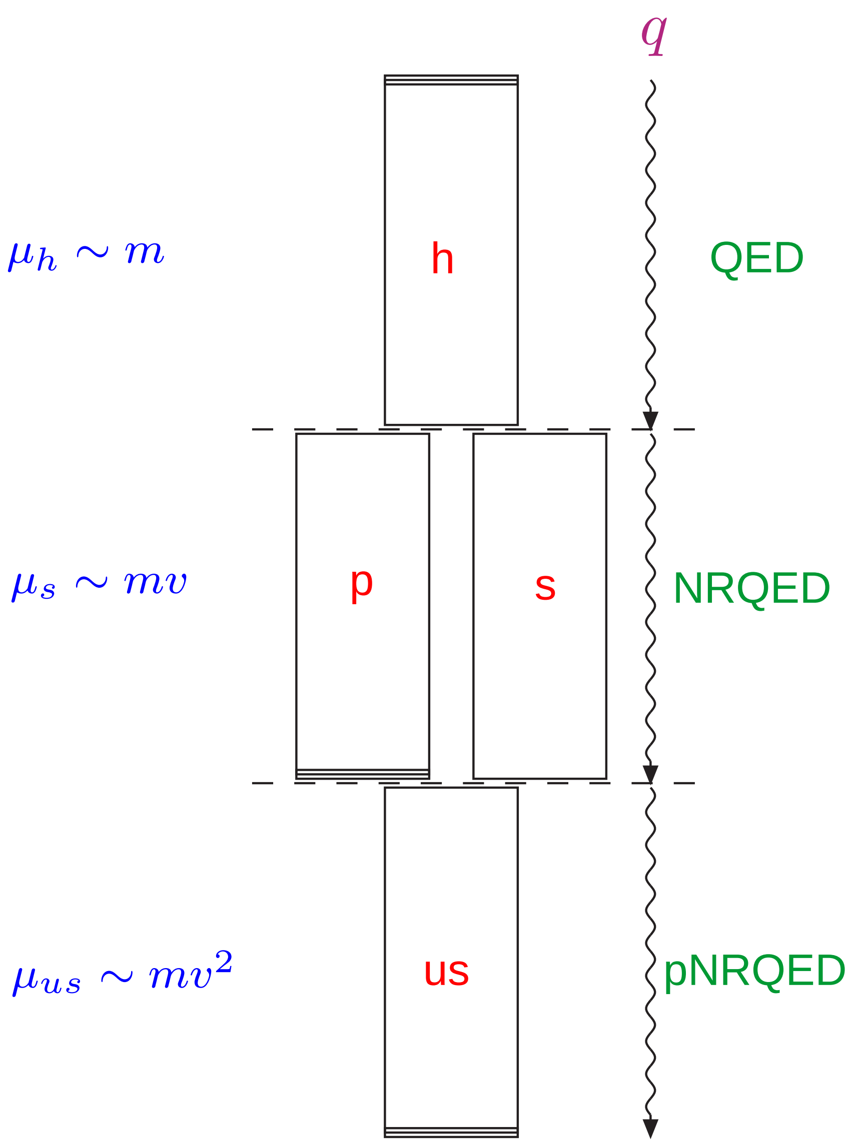
  potential fermions: $p^\mu = (p^0, \vec{p}) \sim (m\,v^2, mv)$

  ultrasoft photons/gluons: $p^\mu = (p^0, \vec{p}) \sim (m\,v^2, mv^2)$
- we want to infer from QED/QCD how these d.o.f. interact
- we will see: the interaction can be described by a potential V (interaction local in t but non-local in $\vec{x}$) and explicit ultrasoft photon/gluon interactions (retardation effects)
- this effective theory is called potential NRQED (pNRQED) and $\mathcal{L}_{\rm pNRQED}(\psi_p, A_{\rm us})$
- we will get there by going through another ET, NRQED with the following additional d.o.f:

  soft particles: $p^\mu = (p^0, \vec{p}) \sim (m\,v, mv)$

  potential photons/gluons: $p^\mu = (p^0, \vec{p}) \sim (m\,v^2, mv^2)$
- NRQED is a local theory (in t and $\vec{x}$) that is obtained by integrating out hard modes from QED
- matching coefficients evaluated at hard scale, then using rgi evolved to soft scale
  $\Longrightarrow$ resummation of $\log \mu_s/\mu_h \sim \log v \sim \log \alpha$

Structure of non-relativistic QED/QCD



underlying theory

$$\mathcal{L}_{\rm QED}(\psi_h, \psi_s, \psi_p, A_h^\mu, A_s^\mu, A_p^\mu, A_{us}^\mu)$$

effective theory `I`

[Caswell, Bodwin, Braaten, Lepage]

$$\mathcal{L}_{\rm NRQED}(\psi_s, \psi_p, A_s^\mu, A_p^\mu, A_{us}^\mu)$$

effective theory `II` (Quantum Mechanics)

[Pineda, Soto]

$$\mathcal{L}_{\rm pNRQED}(\psi_p, A_{us}^\mu)$$

- match free QED Lagrangian $\mathcal{L}^{(0)}_{\rm QED} = \bar\Psi(iD^\mu\gamma_\mu - m)\Psi$ to NRQED counterpart
- introduce separate fields for annihilating electrons ψ and creating positrons χ: $\Psi = \psi + \chi$
- expand in $p/m \sim v$ spinors $u(p)$ (and $v(p)$) in momentum space, $E = \sqrt{\vec{p}^{\,2} + m^2}$

$$u(\vec{p}) = \begin{pmatrix} \sqrt{\frac{E+m}{2E}}\,\xi \\ \frac{\vec\sigma\cdot\vec p}{\sqrt{2E(E+m)}}\,\xi \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{\vec{p}^{\,2}}{8m^2} + \frac{11\vec{p}^{\,4}}{128m^4}\right)\xi \\ \left(\frac{1}{2m} - \frac{3\vec{p}^{\,2}}{16m^2} + \frac{31\vec{p}^{\,4}}{256m^4}\right)\vec\sigma\cdot\vec p\,\xi \end{pmatrix} + \mathcal{O}\left(\frac{1}{m^6}\right)$$

- expand in $p/m \sim v$:

$$\bar u(\vec p)(\not p - m)u(\vec p) = \left(E - m - \frac{p^{\,2}}{2m} + \frac{p^{\,4}}{8m^3}\right)\xi^\dagger\xi + \mathcal{O}\left(\frac{1}{m^4}\right)$$

- free non-relativistic Lagrangian

$$\mathcal{L}^{(0)}_{\rm NRQED} = \psi^\dagger\left(i\,\partial_0 + \frac{\vec\nabla^2}{2m} + \frac{\vec\nabla^4}{8m^3}\right)\psi + \chi^\dagger\left(i\,\partial_0 - \frac{\vec\nabla^2}{2m} - \frac{\vec\nabla^4}{8m^3}\right)\chi + \mathcal{O}\left(\frac{1}{m^4}\right)$$

$\psi^\dagger\vec\nabla^4\psi \sim O_j$ and $1/(8m^3) \sim c_j$

- including interactions $\mathcal{L}_{\rm QED}^{\rm int} = e\,\bar\Psi A^0\gamma^0\Psi - e\,\bar\Psi\vec{A}\cdot\vec\gamma\Psi$
- from gauge invariance we could anticipate $\partial_0 \to \partial_0 - ie\,A^0$ and $\vec\nabla \to \vec\nabla + ie\,\vec{A}$
- here we stubbornly follow matching procedure
  note: $\mathcal{L}_{\rm UT}$ is gauge invariant and all our operators O_j in $\mathcal{L}_{\rm UT}$ are gauge invariant
  $\Longrightarrow$ the c_j must be gauge invariant as well
- then with $\vec q = \vec p\,' - \vec p$ we get (and similar for $\bar v(\vec p\,')$ and $v(\vec p)$)

$$\bar u(\vec p\,')\gamma^0 u(\vec p) = \left(1 - \frac{\vec q^{\,2}}{8m^2}\right)\xi^\dagger\xi + \frac{i}{4m^2}\xi^\dagger\vec\sigma\cdot(\vec p\,'\times\vec p)\xi + \mathcal{O}\left(\frac{1}{m^3}\right)$$

$$\bar u(\vec p\,')\vec\gamma\, u(\vec p) = \frac{1}{2m}\xi^\dagger\left((\vec p + \vec p\,') + i(\vec\sigma\times\vec q)\right)\xi + \mathcal{O}\left(\frac{1}{m^3}\right)$$

- the interaction part of the non-relativistic Lagrangian

$$\mathcal{L}_{\rm NRQED}^{\rm int} = e\,A^0\,\psi^\dagger\psi - \frac{e}{2m}\psi^\dagger\vec A\cdot(\vec p + \vec p\,')\psi - \frac{e}{8m^2}A^0\,\psi^\dagger\vec q\,'^2\psi$$
$$+\frac{i\,e}{4m^2}A^0\,\psi^\dagger\vec\sigma\cdot(\vec p\,'\times\vec p)\psi - \frac{i\,e}{2m}\psi^\dagger\vec A\cdot(\vec\sigma\times\vec q)\psi + \chi\text{-terms} + \mathcal{O}\left(\frac{1}{m^3}\right)$$

combine:

$$\mathcal{L}_{\rm NRQED} = \psi^\dagger \left(i\,\partial_0 + \frac{\vec{\nabla}^2}{2m} + \frac{\vec{\nabla}^4}{8m^3} \right) \psi \; + e\,A^0\,\psi^\dagger\psi \; - \frac{e}{2m}\,\psi^\dagger \vec{A}\cdot(\vec{p}+\vec{p}\,')\psi$$

$$- \frac{e}{8m^2}\,A^0\,\psi^\dagger \vec{q}\,'^2\psi \; + \; \frac{i\,e}{4m^2} A^0\,\psi^\dagger \vec{\sigma}\cdot(\vec{p}\,'\times\vec{p})\psi - \frac{i\,e}{2m}\,\psi^\dagger \vec{A}\cdot(\vec{\sigma}\times\vec{q})\psi$$

$$= \psi^\dagger \left(i\,D_0 + \frac{\vec{D}^2}{2m} + \frac{\vec{D}^4}{8m^3} \right) \psi - \frac{e}{2m}\,\psi^\dagger \vec{\sigma}\cdot\vec{B}\,\psi + \frac{e}{8m^2}\,\psi^\dagger(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D})\psi$$

$$+ \frac{ie}{8m^2}\,\psi^\dagger \vec{\sigma}\cdot(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})\psi \; + \; \chi\text{-terms} + \mathcal{O}\left(\frac{1}{m^4}\right)$$

with $E^i = F^{i0}$ and $B^i = -1/2\,\epsilon^{ijk} F_{jk}$ or

$$\vec{E} = -\vec{\nabla}(A^0) - \partial^0\vec{A} - ig\left[T^b, T^c\right]\vec{A}^{\,b}(A^0)^c \quad \text{and} \quad \vec{B} = \vec{\nabla}\times\vec{A} - \frac{ig}{2}\left[T^b, T^c\right]\vec{A}^{\,b}\times\vec{A}^{\,c}$$

note: all operators are gauge independent!
even in non-abelian case

$$\vec{E}^{\,a} \to \vec{E}^{\,a} + f^{abc}\vec{E}^{\,b}\omega^c$$
$$\vec{B}^{\,a} \to \vec{B}^{\,a} + f^{abc}\vec{B}^{\,b}\omega^c$$

going from QED to QCD and preparing for loops

loop calculations to be done in D dimensions (dimensional regularization):

avoid intrinsic 4-dim objects like ϵ^{ijk}, $\times$ etc.

$$\mathcal{L}_{\rm NRQCD} = \psi^\dagger \left(i\, D_0 + \frac{\vec{D}^2}{2m} + \frac{\vec{D}^4}{8m^3} \right) \psi - \frac{c_F\, g}{2m}\, \psi^\dagger \left(\frac{-\sigma^{ij} F^{ij}}{2} \right) \psi + \frac{c_D\, g}{8m^2}\, \psi^\dagger \left[D^i, E^i \right] \psi$$

$$+ \frac{c_s\, ig}{8m^2}\, \psi^\dagger \sigma^{ij} \left[D^i, E^j \right] \psi + \mathcal{L}_{\rm light} + \chi\text{-terms} + \mathcal{O}\left(\frac{1}{m^4} \right)$$

define D-dimensional Pauli “algebra”:

$$\sigma^{ij} = \frac{[\sigma^i, \sigma^j]}{2\, i} \quad \stackrel{D\to 4}{\longrightarrow} \quad \epsilon^{ijk} \sigma^k$$

$$\frac{-\sigma^{ij} F^{ij}}{2} \quad \stackrel{D\to 4}{\longrightarrow} \quad \vec{\sigma} \cdot \vec{B}$$

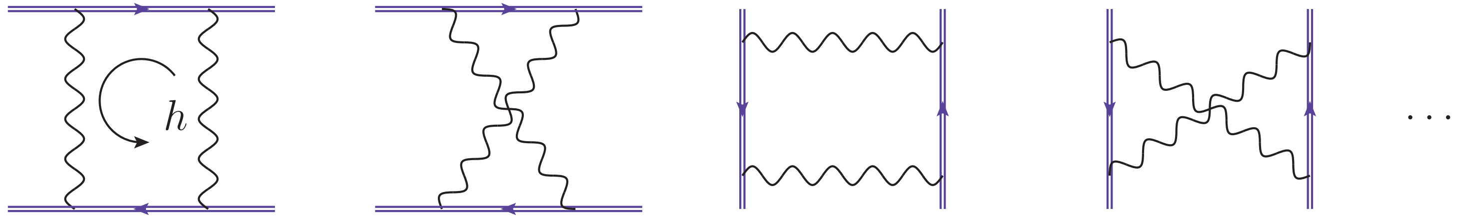
matching coefficients:

$$c_i(\mu_h) = 1 + \alpha_s \left(\log(\mu_h/m) + \text{cst} \right) + \mathcal{O}(\alpha_s^2)$$

contain effects of hard modes

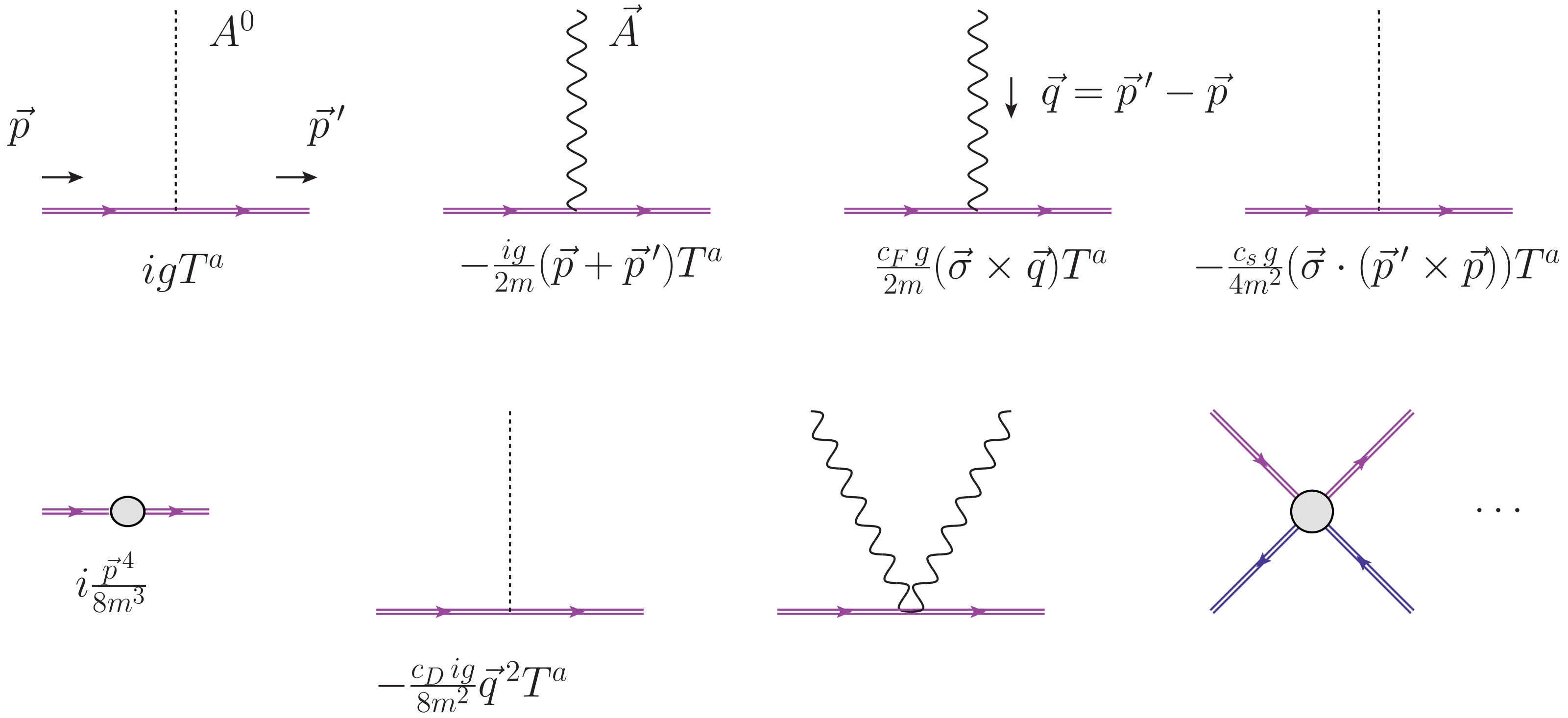
At $\mathcal{O}(1/m^2)$ there are also four-fermion operators

$$\delta\mathcal{L}_{\rm NRQCD} = \frac{d_{ss}}{m_1\, m_2}\, \psi_1^\dagger \psi_1\ \chi_2^\dagger \chi_2 + \frac{d_{sv}}{m_1\, m_2}\, \psi_1^\dagger \vec{\sigma} \psi_1\ \chi_2^\dagger \vec{\sigma} \chi_2 + \frac{d_{vs}}{m_1\, m_2}\, \psi_1^\dagger T^a \psi_1\ \chi_2^\dagger T^a \chi_2 + \frac{d_{vv}}{m_1\, m_2}\, \psi_1^\dagger \vec{\sigma} T^a \psi_1\ \chi_2^\dagger \vec{\sigma} T^a \chi_2$$



- effects of hard loops are encoded in matching coefficients $d \sim \mathcal{O}(\alpha)$
- compare “standard” BSM effective operators

- we have now a theory with new Feynman rules



- this theory reproduces QED/QCD Green functions in the non-relativistic limit up to the order to which the matching has been done

- expansion in $\sim p/m \sim v$ is trivial (if tedious) at tree level
- how to expand in loops ?
    - loop momentum k runs through all scales $0 \to m\,v^2 \to m\,v \to m \to \infty$
    - computing full integral and then expanding is neither efficient nor systematic (power counting)
- method of regions (expand before doing the integration)
    - separate expansion of integrand in all regions
    - sum of all regions add up to full result
    - each part is simpler and has unique power counting
    - identify modes [Beneke, Smirnov] $\Rightarrow$ asymptotic expansion (method of regions)

hard	$k^\mu \sim m$	
soft	$k^\mu \sim mv$	expand integrand not integral
potential	$k^0 \sim mv^2;\ \vec{k} \sim mv$	
ultrasoft	$k^\mu \sim mv^2$	

- $\int d^D k\, f(k,p,m) = \int d^D k\, f_{\rm h} + \int d^D k\, f_{\rm p} + \int d^D k\, f_{\rm s} + \int d^D k\, f_{\rm us}$

Method of regions: a simple example

Let $p^2 \ll M^2$ and assume we want to compute (the first few terms in an expansion in $p^2/M^2 \ll 1$ of) the integral (pick μ s.t. $p^2 \ll \mu^2 \ll M^2$)

$$\int \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

Method of regions: a simple example

Let $p^2 \ll M^2$ and assume we want to compute (the first few terms in an expansion in $p^2/M^2 \ll 1$ of) the integral (pick μ s.t. $p^2 \ll \mu^2 \ll M^2$)

$$\int \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

$$= \int_{|k|<\mu} \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)} + \int_{|k|>\mu} \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

Method of regions: a simple example

Let $p^2 \ll M^2$ and assume we want to compute (the first few terms in an expansion in $p^2/M^2 \ll 1$ of) the integral (pick μ s.t. $p^2 \ll \mu^2 \ll M^2$)

$$\int \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

$$= \int_{|k|<\mu} \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)} + \int_{|k|>\mu} \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

$$= \int_{|k|<\mu} \frac{d^d k}{(k^2 - p^2)} \sum_n \frac{(k^2)^n}{(M^2)^{n+1}} + \int_{|k|>\mu} \frac{d^d k}{(k^2 - M^2)} \sum_n \frac{(p^2)^n}{(k^2)^{n+1}}$$

Method of regions: a simple example

Let $p^2 \ll M^2$ and assume we want to compute (the first few terms in an expansion in $p^2/M^2 \ll 1$ of) the integral (pick μ s.t. $p^2 \ll \mu^2 \ll M^2$)

$$\int \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

$$= \int_{|k|<\mu} \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)} + \int_{|k|>\mu} \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

$$= \int_{|k|<\mu} \frac{d^d k}{(k^2 - p^2)} \sum_n \frac{(k^2)^n}{(M^2)^{n+1}} + \int_{|k|>\mu} \frac{d^d k}{(k^2 - M^2)} \sum_n \frac{(p^2)^n}{(k^2)^{n+1}}$$

$$= \int \frac{d^d k}{(k^2 - p^2)} \sum_n \frac{(k^2)^n}{(M^2)^{n+1}} + \int \frac{d^d k}{(k^2 - M^2)} \sum_n \frac{(p^2)^n}{(k^2)^{n+1}} + \text{tadpoles}$$

Method of regions: a simple example

Let $p^2 \ll M^2$ and assume we want to compute (the first few terms in an expansion in $p^2/M^2 \ll 1$ of) the integral (pick μ s.t. $p^2 \ll \mu^2 \ll M^2$)

$$\int \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

$$= \int_{|k|<\mu} \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)} + \int_{|k|>\mu} \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

$$= \int_{|k|<\mu} \frac{d^d k}{(k^2 - p^2)} \sum_n \frac{(k^2)^n}{(M^2)^{n+1}} + \int_{|k|>\mu} \frac{d^d k}{(k^2 - M^2)} \sum_n \frac{(p^2)^n}{(k^2)^{n+1}}$$

$$= \underbrace{\int \frac{d^d k}{(k^2 - p^2)} \sum_n \frac{(k^2)^n}{(M^2)^{n+1}} + \int \frac{d^d k}{(k^2 - M^2)} \sum_n \frac{(p^2)^n}{(k^2)^{n+1}}}_{\text{additional UV – IR singularities possible}}$$

Method of regions: a simple example

Let $p^2 \ll M^2$ and assume we want to compute (the first few terms in an expansion in $p^2/M^2 \ll 1$ of) the integral (pick μ s.t. $p^2 \ll \mu^2 \ll M^2$)

$$\int \frac{d^d k}{(k^2 - p^2)(k^2 - M^2)}$$

$$= \underbrace{\int \frac{d^d k}{(k^2 - p^2)} \sum_n \frac{(k^2)^n}{(M^2)^{n+1}}}_{\text{soft}} + \underbrace{\int \frac{d^d k}{(k^2 - M^2)} \sum_n \frac{(p^2)^n}{(k^2)^{n+1}}}_{\text{hard}}$$

- identify modes: soft $(k \sim p)$ and hard $(k \sim M)$ (in general more)
- expand integrand in each region to whatever order required
- each term has a well-defined scaling in $p^2/M^2 \;\rightarrow$ power counting
- no explicit cutoff needed (dimensional regularization is important)

Method of regions: a simple example

$$\int \frac{d^d k}{(k^2 - p^2)^a (k^2 - M^2)^b}$$

$$= \frac{i(-1)^{a+b}}{(4\pi)^{d/2}} \left(M^2\right)^{\frac{d}{2}-a-b} \frac{\Gamma(a+b-\frac{d}{2})}{\Gamma(a+b)} \, {}_2F_1\left({a; a+b-\frac{d}{2} \atop a+b} \middle| 1 - \frac{p^2}{M^2} \right)$$

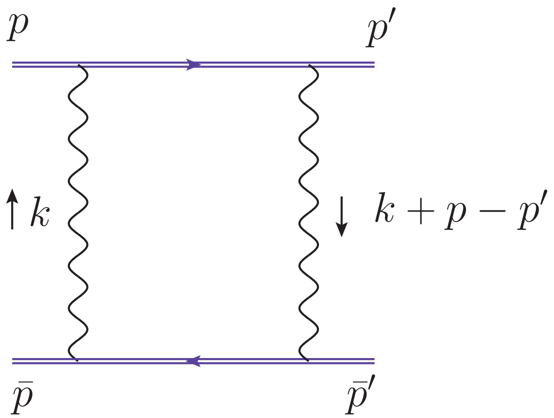
$$= \sum_{n=0}^{\infty} \frac{(-n-b+1)_n}{\Gamma(n+1)} \left(-M^2\right)^{-b-n} \int \frac{d^d k}{(k^2)^{-n} (k^2 - p^2)^a}$$

$$+ \sum_{n=0}^{\infty} \frac{(-n-a+1)_n}{\Gamma(n+1)} \left(-p^2\right)^{n} \int \frac{d^d k}{(k^2)^{a+n} (k^2 - M^2)^b}$$

$$= \frac{i(-1)^{a}}{(4\pi)^{d/2}} \left(p^2\right)^{\frac{d}{2}-a} \left(-M^2\right)^{-b} \frac{\Gamma(a-\frac{d}{2})}{\Gamma(a)} \, {}_2F_1\left({\frac{d}{2};\ b \atop 1-a+\frac{d}{2}} \middle| \frac{p^2}{M^2} \right)$$

$$+ \frac{i(-1)^{a+b}}{(4\pi)^{d/2}} \left(M^2\right)^{\frac{d}{2}-a-b} \frac{\Gamma(\frac{d}{2}-a)\Gamma(a+b-\frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})} \, {}_2F_1\left({a; a+b-\frac{d}{2} \atop 1+a-\frac{d}{2}} \middle| \frac{p^2}{M^2} \right)$$

example of hard loop



before expansion

$$I_{\rm full} = \int \frac{d^D k}{k^2\,[(k+p)^2 - m_1^2]\,(k+p-p')^2\,[(k-\bar{p})^2 - m_2^2]}$$

after expansion

$$I_{\rm h} = \int \frac{d^D k}{k^2\,[k^2 - m_2^2]\,k^2\,[k^2 - m_1^2]}$$

- $I_{\rm h}$ is much simpler
- $I_{\rm full}$ and $I_{\rm h}$ have the same UV-singularities $\Longrightarrow$ renormalization
- $I_{\rm h}$ has IR singularities not present in $I_{\rm full}$ $\Longrightarrow$ canceled by UV singularities of ET
- scaling in v: $I_{\rm h} \sim 1$ (known before integration) $k \sim m \sim 1$
- scaling in v: $I_{\rm full}$ not uniform (different scales) $p_0 \sim mv^2$, $p \sim mv^2$, $k \sim$ anything

renormalization group improvement

- explicit computation of matching coefficients at one-loop after UV renormalization typically yields $c_i(\mu) = 1 + \alpha(\mu)\left(\gamma_i^0 \left[\frac{1}{\epsilon} - \log\frac{m}{\mu}\right] + \#\right)$
- the singularity is cancelled by a UV singularity of NRQCD (anomlaous dimension γ_i of NRQCD operators)
- the hard matching coefficient has to be computed at a hard scale $\mu \to \mu_h \sim m$ to avoid large logs
- when used in NRQCD it has to be evaluated at the soft scale $\mu \to \mu_s \sim m\,v$
- solution to standard rge for anomlaous dimension $\mu\frac{d}{d\mu}c_i(\mu) = \gamma_i c_i(\mu)$ is given by

  $$c_i(\mu_s) = c_i(\mu_h) \exp\int_{\alpha(\mu_s)}^{\alpha(\mu_h)} \frac{\gamma_i(\alpha)\,d\alpha}{2\,\beta(\alpha)}$$
- this resums all (potentially large) logarithms $L \equiv \log\mu_h/\mu_s \sim \log\alpha \sim \log v$
- with γ_i^0 we get NLL (next-to-leading logarithmic) accuracy, i.e. $\alpha^n\,L^{n-1}$

NRQCD Lagrangian [Caswell, Bodwin, Braaten, Lepage]

but now in $D = 4 - 2\epsilon$ dimensions

$$
\begin{aligned}
\mathcal{L}_{\rm NRQCD} \;\; = \;\; & \psi^\dagger \left(i D^0 + c_k \frac{\vec{D}^2}{2m} \right) \psi + \frac{c_4}{8m^3} \psi^\dagger \vec{D}^4 \psi - \frac{g\, c_F}{2m} \psi^\dagger \sigma^i B^i \psi \\
+ \;\; & \frac{g\, c_D}{8m^2} \psi^\dagger \left[D^i, E^i \right] \psi + \frac{i g\, c_S}{8m^2} \psi^\dagger \sigma^{ij} \left[D^i, E^j \right] \psi + (\psi \leftrightarrow \chi) \\
+ \;\; & \frac{\alpha_s\, d_{ss}}{m^2} \psi^\dagger \psi\, \chi^\dagger \chi + \frac{\alpha_s\, d_{sv}}{m^2} \psi^\dagger \sigma^i \psi\, \chi^\dagger \sigma^i \chi \\
+ \;\; & \frac{\alpha_s\, d_{vs}}{m^2} \psi^\dagger T^a \psi\, \chi^\dagger T^a \chi + \frac{\alpha_s\, d_{vv}}{m^2} \psi^\dagger \sigma^i T^a \psi\, \chi^\dagger \sigma^i T^a \chi + \mathcal{L}_{\rm light}
\end{aligned}
$$

- resum $\ln(\mu_h/\mu_s) = \ln v$ in c_i and d_{ij} using renormalization group
- RGI: single heavy quark sector as in HQET [Bauer, Manohar, ...]
  RGI: four heavy quark operators [Pineda]

- QED $\to$ NRQED: hard loops $p^\mu \sim m$ integrated out, not dynamical any longer (we exploited $m \gg mv$)
- we are left with

soft	$p^\mu \sim mv$
potential	$p^0 \sim mv^2;\ \vec{p} \sim mv$
ultrasoft	$p^\mu \sim mv^2$

- an operator like $\psi^\dagger \left[D^i, E^i\right] \psi$ does not have a fixed power in v
- final state has only potential fermions and ultrasoft photons
- NRQED $\to$ potential NRQED (pNRQED):
  integrate out soft fermions and potential and soft photons
- in pNRQED only potential fermions and ultrasoft photons are dynamical (exploit also $mv \gg mv^2$)
- “integrating out” technically again with method of regions

After a few slides we will end up with the pNRQCD Lagrangian in $d=3-2\epsilon$ dimensions

$$
\begin{aligned}
\mathcal{L}_{\rm QED} \quad &\Rightarrow \quad \mathcal{L}_{\rm NRQED} \;\Rightarrow \\
\mathcal{L}_{\rm pNRQCD} \quad &= \quad \psi^\dagger \left(iD^0 + \frac{\partial^2}{2m}\right)\psi + \chi^\dagger \left(iD^0 - \frac{\partial^2}{2m}\right)\chi \\
&+ \quad \int d^3 r \left(\psi^\dagger T^a \psi\right) V \left(\chi^\dagger T^a \chi\right) \\
&+ \quad \psi^\dagger \left(\frac{\partial^4}{8m^3} - g_s\, \vec{x}\cdot\vec{E}\right)\psi + \chi^\dagger \left(-\frac{\partial^4}{8m^3} - g_s\, \vec{x}\cdot\vec{E}\right)\chi \\
V \quad &= \quad -4\pi C_F \frac{\alpha_s}{\vec{q}^{\,2}} - C_F \frac{\alpha_s^2}{\vec{q}^{\,2}} \left(a_1 - \beta_0 \ln \frac{\vec{q}^{\,2}}{\mu^2}\right) + \ldots \\
&- \quad C_F C_A \alpha_s^2\, D_s^{(1)} \frac{\pi^2\, \mathcal{K}(\epsilon)}{m\, q^{1+2\epsilon}} + \frac{3\pi C_F \alpha_s\, D_{d,s}^{(2)}}{m^2} - \frac{4\pi C_f\, D_{s^2}^{(2)}}{dm^2} [s_1^i, s_1^j][s_2^i, s_2^j] \ldots
\end{aligned}
$$

- static potential (known to a_3), non-analytic potential . . ., d-dim generalization of Breit-Fermi potential (with spin-spin, L^2 etc)
- resum $\ln(\mu_s/\mu_{us}) = \ln v$ in matching coefficients $D_s^{(1)}, D_{d,s}^{(2)}, D_{s^2}^{(2)} \ldots$

Power counting

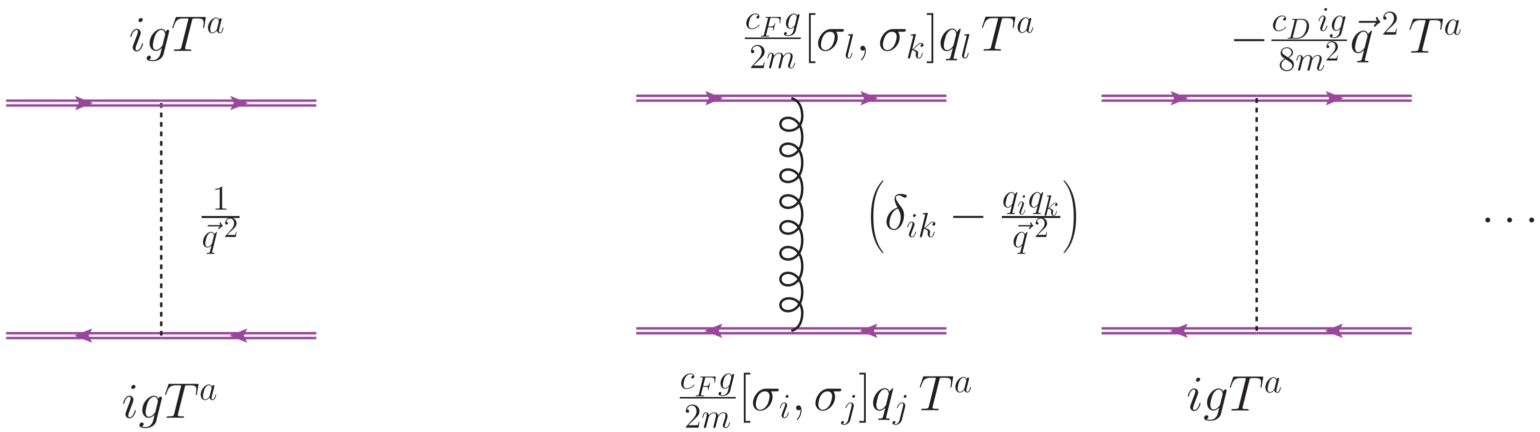
	mom	prop form	prop.	d^4k	field
pot. Q	$(v^2, \vec{v})$	$[k^0 - \vec{k}^{\,2}/(2m)]^{-1}$	v^{-2}	v^5	$v^{3/2}$
pot. g		$[-\vec{k}^{\,2}]^{-1}$	v^{-2}	v^5	$v^{3/2}$
soft Q	$(v, \vec{v})$	$[k^0]^{-1}$	v^{-1}	v^4	$v^{3/2}$
soft g		$[k^2]^{-1}$	v^{-2}	v^4	v
us g	$(v^2, \vec{v}^{\,2})$	$[k^2]^{-1}$	v^{-4}	v^8	v^2

$$\langle \psi(x)\psi(0)\rangle = \int \frac{d^4k}{k^2}$$

operators in $\mathcal{L}_{\rm pNRQCD}$

$\psi^\dagger \left(i\partial^0 + (\partial^2/2m)\right)\psi$	$v^{3/2}\, v^2\, v^{3/2} = v^5$	LO
$\left(\psi^\dagger T^a \psi\right)\,(\alpha_s/\vec{q}^{\,2})\,\left(\chi^\dagger T^a \chi\right)$	$v^3\,(\alpha_s/v^2)\,v^3 = \alpha_s v^4$	LO
$\left(\psi^\dagger T^a \psi\right)\,(\alpha_s^2/\vec{q}^{\,2})\,\left(\chi^\dagger T^a \chi\right)$	$v^3\,(\alpha_s^2/v^2)\,v^3 = \alpha_s^2 v^4$	NLO
$\left(\psi^\dagger T^a \psi\right)\,(\alpha_s^2/q)\,\left(\chi^\dagger T^a \chi\right)$	$v^3\,(\alpha_s^2/v)\,v^3 = \alpha_s^2 v^3$	NNLO
$\psi^\dagger \left(g_s\,\vec{x}\cdot\vec{E}\right)\psi$	$v^{3/2}\,\sqrt{\alpha_s}v^4\,v^{3/2} = \sqrt{\alpha_s}v^7$	NNNLO

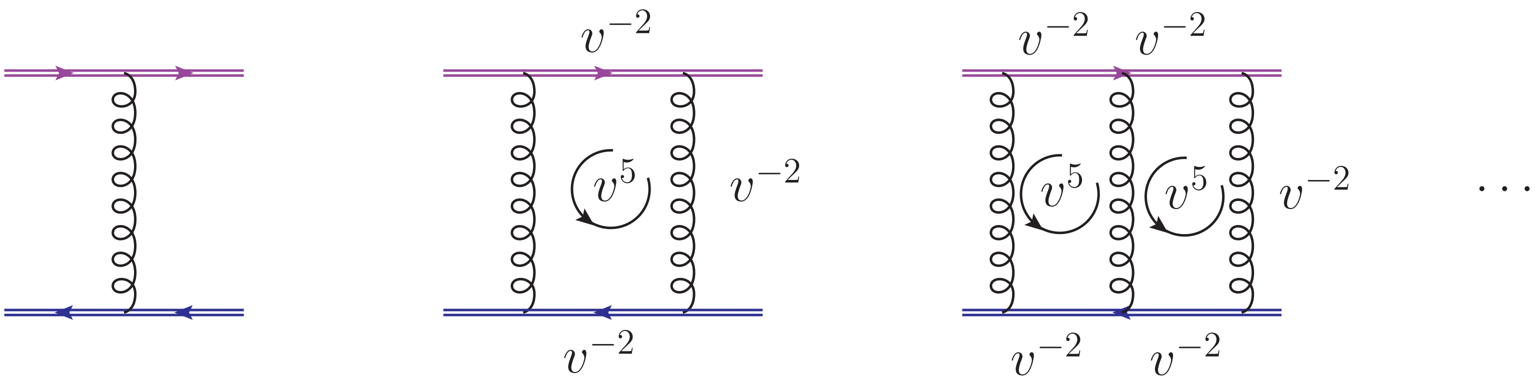
- Breit potential the naive diagram we started with now looks like



LO δV_{Breit}

- the LO potential: $V = \begin{cases} -\frac{\alpha_s}{4\pi}\frac{C_f}{\vec{q}^2} & \text{colour singlet} \\ -\frac{\alpha_s}{4\pi}\frac{C_f - C_A/2}{\vec{q}^2} & \text{colour octet} \end{cases}$
- the Breit potential depends on spin projection
- $\delta V_{\text{Breit}} = \begin{cases} \frac{\vec{p}^2}{m^2} + \frac{\vec{q}^2}{m^2}\left(\frac{(D-2)(D-5)}{4(D-1)}\, c_F^2 - \frac{1}{4}(1+c_D)\right) & \text{spin 1} \\ \frac{\vec{p}^2}{m^2} + \frac{\vec{q}^2}{m^2}\left(\frac{(D-2)}{4}\, c_F^2 - \frac{1}{4}(1+c_D)\right) & \text{spin 0} \end{cases}$

Power counting potential ladder diagrams have to be resummed



α/v^2 $\alpha/v^2\,\alpha/v$ $\alpha/v^2\,(\alpha/v)^2$

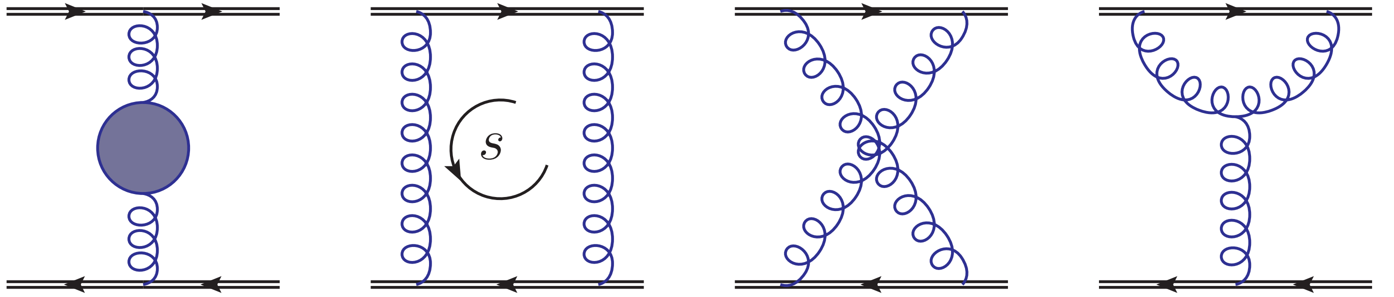
This gives the Green function in momentum space

$$\tilde{G}_c(\vec{p},\vec{p}\,',E) = (2\pi)^d\delta^{(d)}\left(\vec{p}-\vec{p}\,'\right)\frac{-1}{E-\vec{p}^{\,2}/m} + \frac{4\pi C_F\alpha_s}{(E-\vec{p}^{\,2}/m)\,(\vec{p}-\vec{p}\,')^2\,(E-\vec{p}\,'^{\,2}/m)} + \text{finite}$$

or via Fourier in coordinate space ($\nu \equiv C_F\,\alpha_s/(2\sqrt{-E/m})$)

$$G_c(0,0,E) = \frac{\alpha_s\,C_F\,m^2}{8\pi}\left(\frac{1}{2\epsilon} - \ln\frac{-4mE}{\mu^2} - \frac{1}{\nu} - 2\psi(1-\nu) - 2\gamma_E + 1\right)$$

as an example consider the static potential at NLO



- all diagrams taken separately are gauge dependent
- gauge dependence cancels in sum (as it must) $\to a_1$ is gauge independent !!
- consider e.g. box diagram
  - hard loop $\to$ matching coefficient of four-fermion operator
  - potential loop $\to$ LO Green function
  - soft loop $\to$ NLO static potential
- an ordinary QED Feynman diagram splits and contributes to different parts

pNRQCD Lagrangian [Pineda, Soto]

$$\mathcal{L}_{\rm pNRQCD} = \psi^\dagger \left(iD^0 + \frac{\partial^2}{2m}\right)\psi + \chi^\dagger \left(iD^0 - \frac{\partial^2}{2m}\right)\chi$$
$$+ \int d^3r \left(\psi^\dagger T^a \psi\right) V \left(\chi^\dagger T^a \chi\right)$$
$$+ \psi^\dagger \left(\frac{\partial^4}{8m^3} - g_s\,\vec{x}\cdot\vec{E}\right)\psi + \chi^\dagger \left(-\frac{\partial^4}{8m^3} - g_s\,\vec{x}\cdot\vec{E}\right)\chi$$

$$V = -4\pi C_F \frac{\alpha_s}{\vec{q}^{\,2}} + \delta V$$

- QFT $\to$ potential $V^0 + \delta V$
- each term has a well-defined power counting, ultrasoft effects enter at NNNLO
- recall everything you know about QM and do QM pert. theory in momentum space
- for higher-order corrections evaluate single and double insertions

$$\delta G_c(0,0,E) = \int \prod \frac{d^d\vec{p}_i}{(2\pi)^d}\, \tilde{G}_c(\vec{p}_1,\vec{p}_2,E)\,\delta V(\vec{p}_2,\vec{p}_3)\,\tilde{G}_c(\vec{p}_3,\vec{p}_4,E)$$

- all singularities (IR and UV) are consistently treated with dimensional regularization

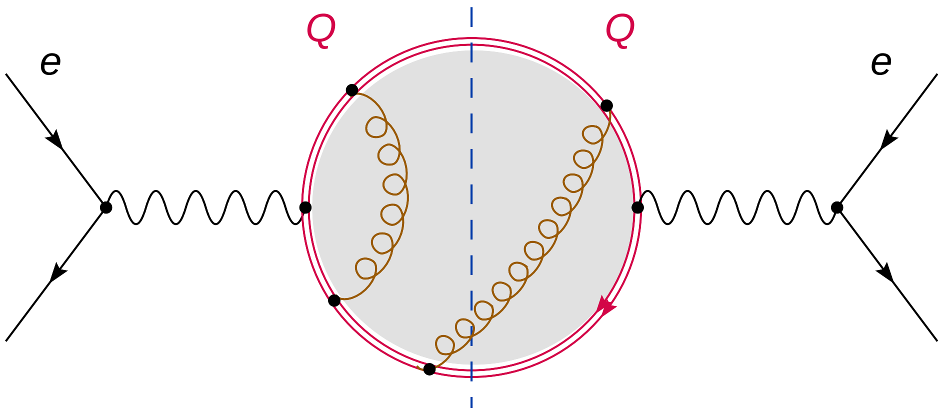
Part II

Applications

Heavy quark pair production: $e^+e^- \to Q\bar{Q}$ $\quad Q \in \{c, b, (t)\}$ $\quad \sqrt{s} \sim 2m$

cross section: $$R_{Q\bar{Q}}(s) \equiv \frac{\sigma(e^+e^- \to Q\bar{Q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi\,\mathrm{Im}\,\{\Pi(s + iO^+)\}$$

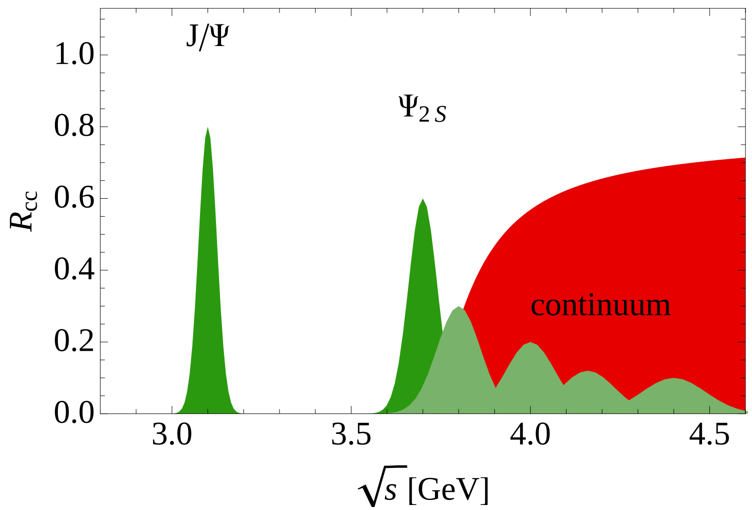
correlator: $$\Pi^{\mu\nu} \equiv i \int d^4x\, e^{i\,qx} \langle 0|T\{j^\mu(x) j^\nu(0)|0\rangle = (-q^2 g^{\mu\nu} + q^\mu\, q^\nu)\,\Pi(q^2)$$

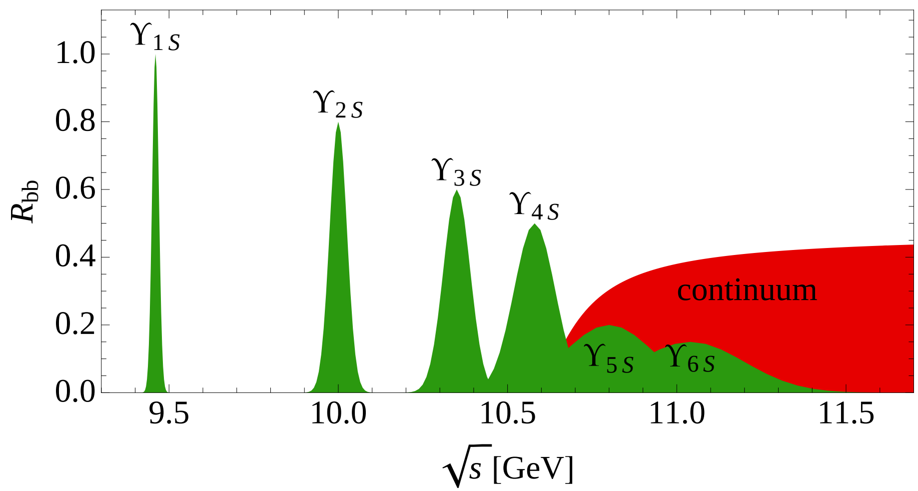


local parton-hadron duality $R_{Q\bar{Q}}$ $\longrightarrow$ global parton-hadron duality M_n

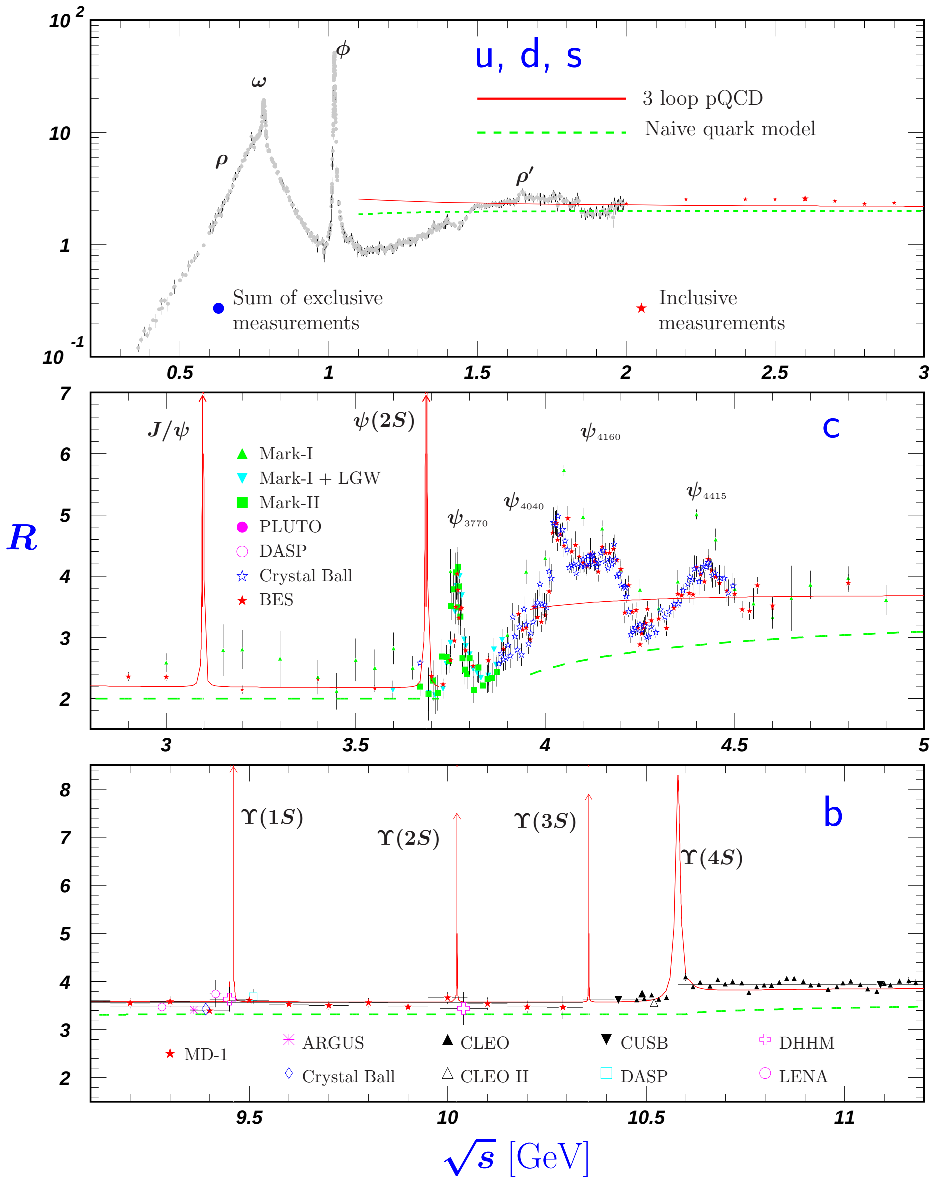
moments :
$$M_n^{\rm th} \equiv \int \frac{ds}{s^{n+1}} R_{Q\bar{Q}}(s) = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2}\right)^n \Pi(q^2)\big|_{q^2=0}$$

$$M_n^{\rm exp} = M_n^{\rm res} + M_n^{\rm cont} = \underbrace{\frac{9\pi}{\alpha_{\rm em}^2} \sum_{k=1}^{K} \frac{\Gamma_k}{M_k^{2n+1}}}_{\text{well known}} + \underbrace{\int_{s \gtrsim s_{\rm thr}} \frac{ds}{s^{n+1}} R_{Q\bar{Q}}(s)}_{\text{poorly known}}$$





in real life:


u, d, s
3 loop pQCD
Naive quark model
ω
ϕ
ρ
ρ'
Sum of exclusive measurements
Inclusive measurements
10^2
10
1
10^{-1}
0.5
1
1.5
2
2.5
3
R
c
J/ψ
$\psi(2S)$
ψ_{3770}
ψ_{4040}
ψ_{4160}
ψ_{4415}
Mark-I
Mark-I + LGW
Mark-II
PLUTO
DASP
Crystal Ball
BES
3.5
4
4.5
5
b
$\Upsilon(1S)$
$\Upsilon(2S)$
$\Upsilon(3S)$
$\Upsilon(4S)$
MD-1
ARGUS
Crystal Ball
CLEO
CLEO II
CUSB
DASP
DHHM
LENA
9.5
10
10.5
11
$\sqrt{s}$ [GeV]

mass of $\Upsilon(nS)$: $M_{\Upsilon(nS)} = 2m_b + E_n$ typical scale: $\mu \sim p \sim \alpha_s C_F m_b/n$

$\mu \sim 1.3\ \mathrm{GeV}$ for $n = 1$

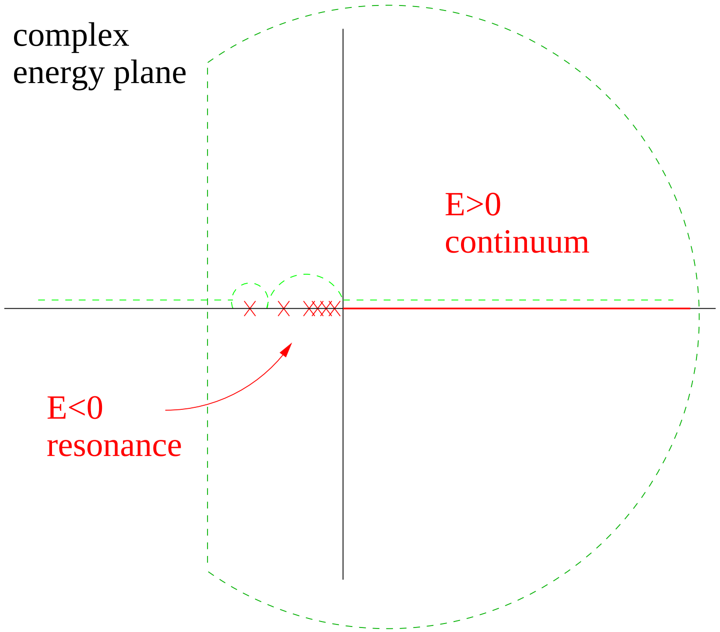
dominant error non-perturbative $\Longrightarrow$ later

moments: $M_n = \int \frac{ds}{s^{n+1}} R_{b\bar{b}}(s)$ typical scale: $\mu \sim 2m_b/\sqrt{n}$

$\mu \sim 2.5\ \mathrm{GeV}$ for $n = 14$

dominant error perturbative

- determination of theoretical moments via integration in complex plane
- typical scale $\mu_s \sim 2m_b/\sqrt{n}$, choose $n \leq 14$
- determine experimental resonanance moments (very well known) and continuum moments (poorly known), choose $n \geq 6$



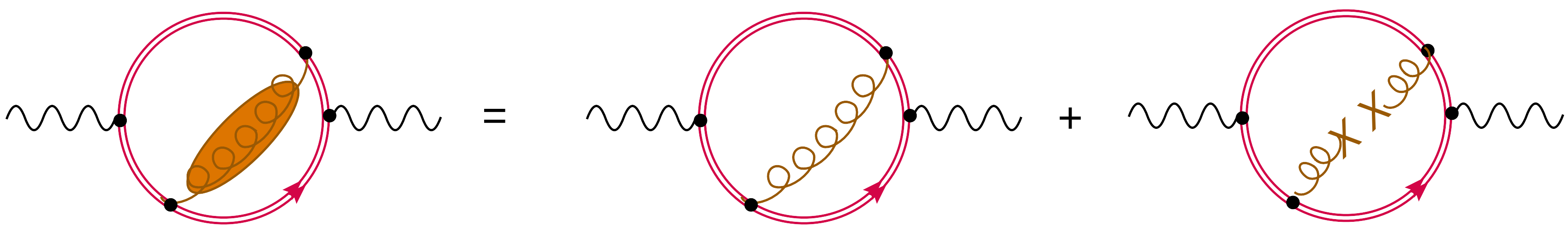
theoretical moments

perturbative part: gluon (quark) propagator $\sim 1/k^2$, but contains terms to all orders in α_s

- in principle well understood
- can be computed with ever increasing accuracy (at the price of running into technical difficulties, current status 4-loop)

non-perturbative part: modification of gluon propagator from $\sim 1/k^2$ for small k^2

- not very well understood $\Rightarrow$ try to minimize the impact of non-perturbative physics
- parametrize ignorance in terms of (ever more suppressed) condensates
- leading contribution from gluon condensate $\langle \frac{\alpha}{\pi} G^2 \rangle$



theory: perturbative part

$$M_n = \int \frac{ds}{s^{n+1}} R_{Q\bar{Q}}(s) \simeq \int \frac{2\,dE}{(2m)^{2n+1}}\, e^{\frac{-n\,E}{m}}\, R_{QQ}(E)$$

relativistic sum rules: n “small”, i.e $n \lesssim 4$ continuum contribution relevant

FO (fixed order) approach

$$\Pi(q^2) = \frac{N_c\, e_Q^2}{(4\pi)^2} \sum_{n\geq 0} C_n \left(\frac{q^2}{4m^2}\right)^n \quad \Longleftrightarrow \quad M_n = \frac{3}{4} N_c\, e_Q^2\, \frac{1}{(2m)^{2n}}\, C_n$$

pole scheme: $C_n^{(\ell)} \sim n^{-3/2}\, (\alpha\sqrt{n})^\ell$

non-relativistic sum rules: n “large”, i.e $n \gtrsim 8$ continuum contribution suppressed

ET (effective theory) approach

define $E = \sqrt{s} - 2m \equiv m v^2$ $\sim$ kinetic energy of heavy quarks if $v \ll 1$

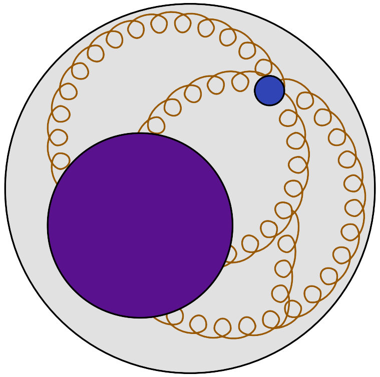
n “large” $\leftrightarrow$ $E \sim m \times 1/n$ and $v \sim 1/\sqrt{n}$ “small” $\Rightarrow$ quantum mechanics large n (non-relativistic) vs small n (relativistic)

- large n corresponds to small $v \sim 1/\sqrt{n}$, conventional fixed order (FO) perturbation theory breaks down (Coulomb singularity), i.e. computing $R_{Q\bar{Q}}$ to α^ℓ we have terms $v\,(\alpha/v)^\ell \sim n^{-1/2}\,(\sqrt{n}\,\alpha)^\ell \quad \longrightarrow$ use effective theory (ET)
- FO: standard expansion in coupling α, keeping full dependence of E, i.e. take into account all powers of $E/m = v^2$
- ET: double expansion in α and $v = \sqrt{E/m}$, using non-relativistic QCD.

R_{QQ}	ET: LO	ET : NLO	ET : NNLO	...
FO: LO	$v\,c_{0,1}$	$v^2\,c_{0,2}$	$v^3\,c_{0,3}$	$v^4\,c_{0,4}$
FO: NLO	$\alpha\,c_{1,0}$	$\alpha\,v\,c_{1,1}$	$\alpha\,v^2\,c_{1,2}$	$\alpha\,v^3\,c_{1,3}$
FO: NNLO	$\alpha^2\,v^{-1}\,c_{2,-1}$	$\alpha^2\,c_{2,0}$	$\alpha^2\,v\,c_{2,1}$	$\alpha^2\,v^2\,c_{2,2}$
⋮	$\alpha^3\,v^{-2}\,c_{3,-2}$	$\alpha^3\,v^{-1}\,c_{3,-1}$	$\alpha^3\,c_{3,0}$	$\alpha^3\,v\,c_{3,1}$

mass schemes So far implicitly understood mass = pole mass m_Q

but pole mass has non-perturbative ambiguity (renormalon) $\Rightarrow$ IR sensitivity $\sim \Lambda_{\rm QCD}$



$$\underbrace{M_{\rm meson}}_{\rm obs} = \underbrace{m_Q}_{\text{pole mass}} + \underbrace{V}_{\rm ambig}$$

For $Q\bar{Q}$ system: m_Q has IR sensitivity, but this cancels in $2m_Q + V_{\rm coul} \simeq M_{\rm meson}$

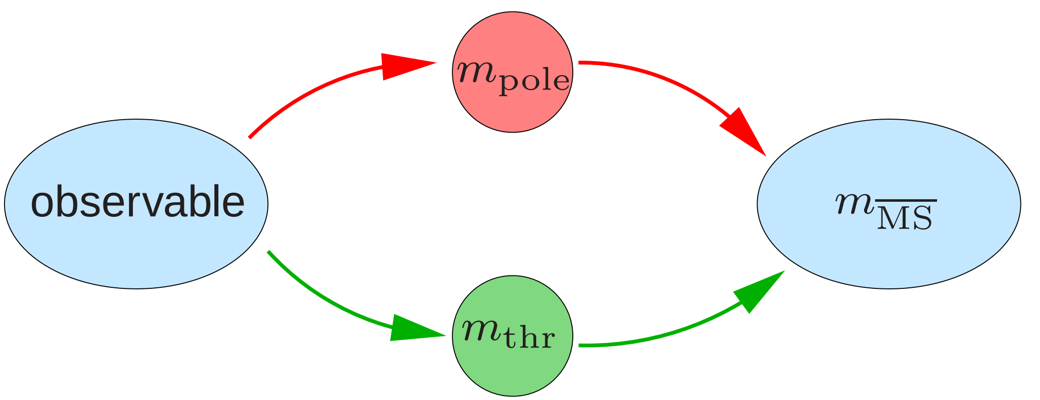
define PS-mass [Beneke] $m_{\rm PS} = m_Q + \frac{1}{2} \int\limits_{q<\mu_F} \frac{d^3\vec{q}}{(2\pi)^3} V_{\rm coul}(q) \quad \text{with} \quad \mu_F \sim m\,v \sim m\,\alpha_s$

other closely related definitions $m_X = m_Q - \delta m$ are possible

these mass definitions are more appropriate for the description of heavy quarks near threshold
$\Rightarrow$ threshold mass

mass schemes

- pole mass is more IR sensitive (renormalon ambiguity) than other mass definitions $\to$ non-perturbative ambiguity $\sim \Lambda_{\rm QCD}$
- use directly $m_{\overline{\rm MS}}$ where possible (relativistic sum rules)
- if use of $m_{\overline{\rm MS}}$ impossible (non-relativistic sum rules) use threshold mass (incoorporates renormalon cancellation) [Bigi et.al; Beneke; Hoang et.al; Pineda]
- express observable in terms of threshold mass (here use PS mass [Beneke] and RS mass [Pineda]) then relate threshold mass to $m_{\overline{\rm MS}}$; (three-loop exact [Melnikov, Ritbergen; Chetyrkin, Steinhauser] and four-loop via large-β_0 approximation)



theory: non-perturbative part

leading contribution from gluon condensate [Shifman et.al; Broadhurst et.al., Ioffe]

$$\delta M_n^{\rm np} = \frac{12\pi^2\, e_Q^2}{(2m_b)^{(2n+4)}}\,\langle\frac{\alpha}{\pi}G^2\rangle\, a_n \left(1+\frac{\alpha}{\pi}\, b_n\right) + \ldots$$

- $a_n \sim n^{3/2}$: importance of non-perturbative effects increases with increasing n
- size of corrections $\frac{\alpha}{\pi}\, b_n$ crucially depends on mass scheme

main questions:

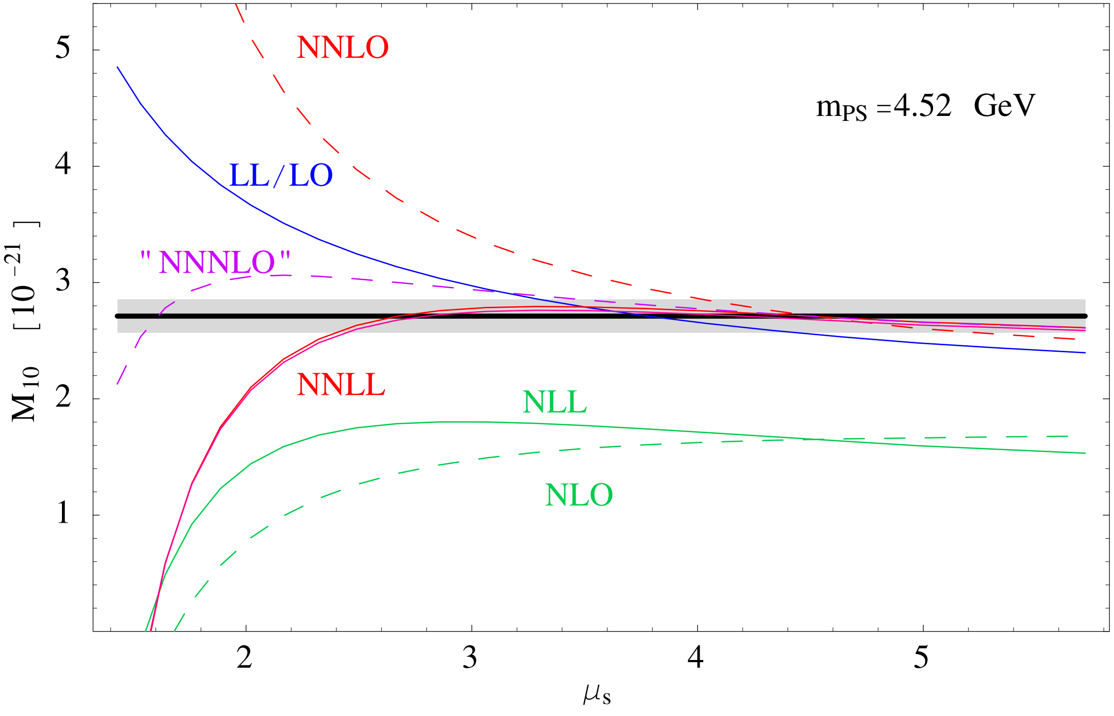
- how important are gluon condensate contributions??
  - $\langle\frac{\alpha}{\pi}G^2\rangle = 0.012~{\rm GeV}^4$ [Shifman et.al. 1978]
  - $\langle\frac{\alpha}{\pi}G^2\rangle = 0.021~{\rm GeV}^4$ [Broadhurst et.al. 1994]
  - $\langle\frac{\alpha}{\pi}G^2\rangle = (0.005 \pm 0.004)~~{\rm GeV}^4$ [Ioffe 2005]
- can we trust the perturbative series of the coefficient function?

common wisdom ??:

- we can ignore $\langle\frac{\alpha}{\pi}G^2\rangle$ contributions in the case of bottom as long as $n \lesssim 16$
- what about the charm case ?

Determination of bottom mass from sum rules take M_{10} as an example:

[Pineda, AS]



consistent results for $6 \leq n \leq 16$

$m_{\rm PS} = 4.52 \pm 0.06$ GeV

convert to $\overline{\rm MS}$:

$\overline{m} = 4.19 \pm 0.06$ GeV

through resummation of $\log v = \log \mu_s/\mu_h$:

- size of corrections reduced
- much improved μ_s scale dependence

} reduced theoretical error

apply to charm ?? $\Rightarrow$ non-perturbative contributions ??

leading contribution from gluon condensate [Shifman et.al; Broadhurst et.al., Ioffe]

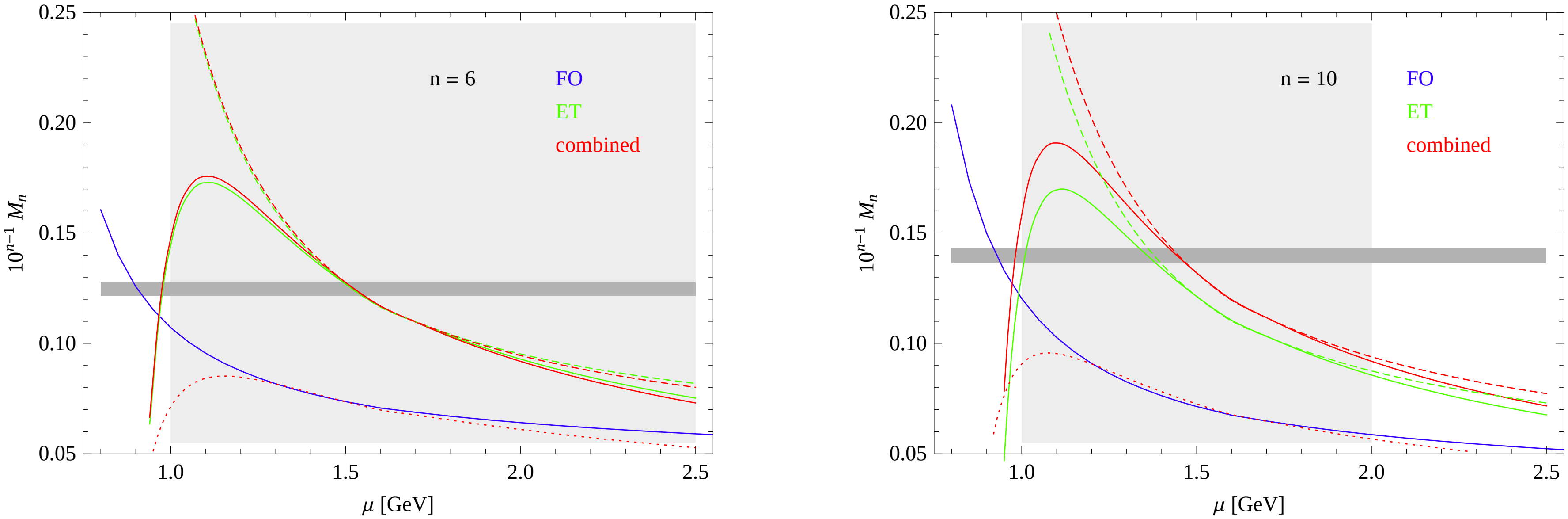
$$\delta M_n^{\rm np} = \frac{12\pi^2\, e_q^2}{(2m_b)^{(2n+4)}} \langle \frac{\alpha}{\pi} G^2 \rangle\, a_n \left(1 + \frac{\alpha}{\pi}\left[b_n - (2n+4)\delta b_X\right]\right) + \ldots \text{ with } a_n \sim n^{3/2}$$

importance of non-perturbative effects increases with increasing n and decreasing m

	n	1	4	8	12	16
bottom	$10^2 \delta M_n^{\rm np}/M_n^{\rm exp}$	-0.003	-0.02	0.02	0.36	1.6
charm	$10^2 \delta M_n^{\rm np}/M_n^{\rm exp}$	0.1	0.7	2.0	3.8	5.9
	$\alpha_s\, b_n^{\rm PS}/\pi$	0.75	0.72	0.56	0.34	0.09

ignore non-perturbative effects and use $n < 16$

Determination of charm mass from sum rules [AS]



n	m	$\delta m^{\rm th}$	$\delta m^{\rm exp}$	δm^{α}	δm^{GG}	δm
3	1508	229	11	41	2	233
6	1506	81	3	27	3	85
10	1503	40	2	19	5	45
16	1500	27	1	14	6	31

"combine":
single moment analysis

$m_{\rm PS} = 1.50 \pm 0.04$ GeV

convert to $\overline{\rm MS}$:

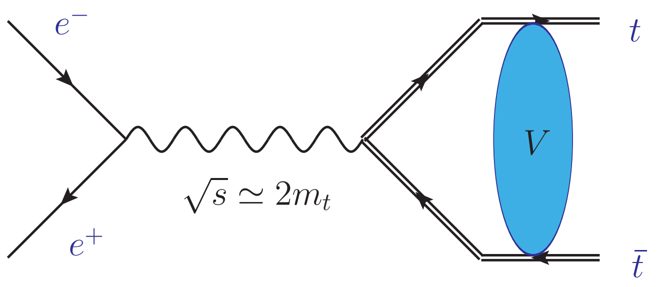
$\overline{m} = 1.25 \pm 0.04$ GeV

Top threshold scan at linear collider

top pair produced near threshold

$E \equiv \sqrt{s} - 2m \ll m$

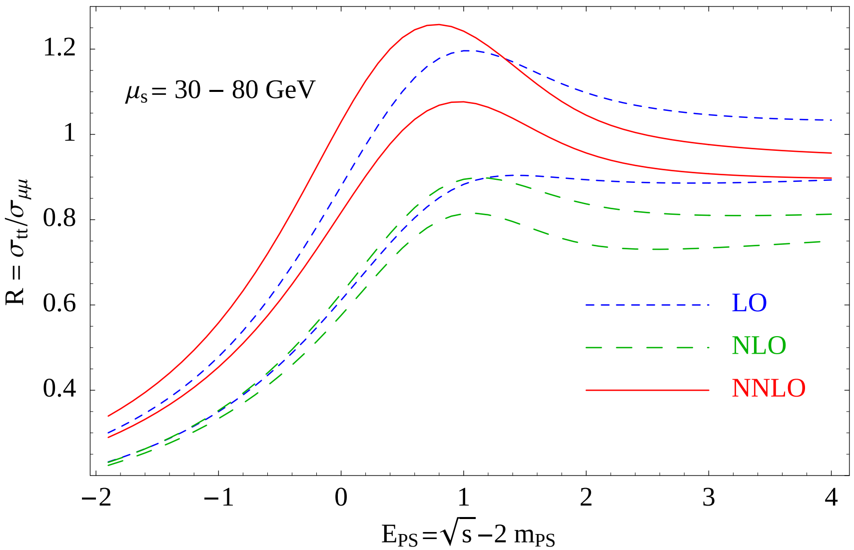
non-relativistic $\rightarrow$ NRQCD



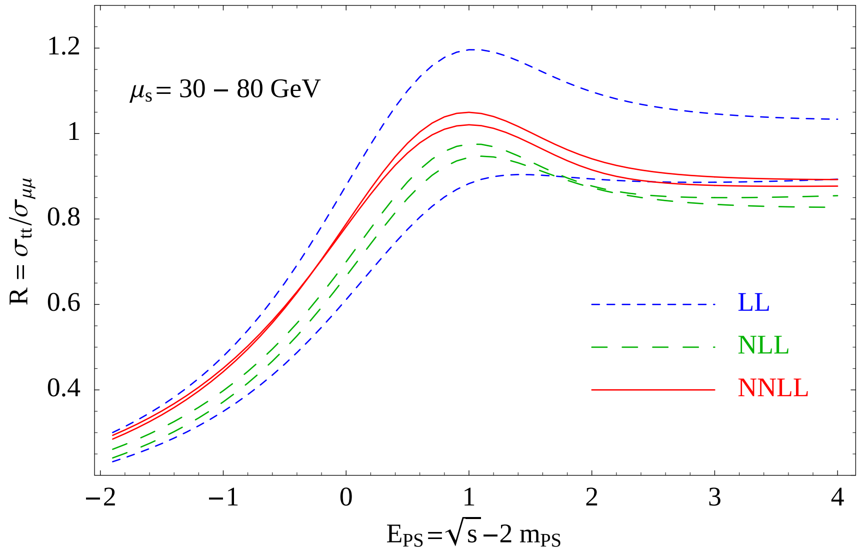
- lifetime for top $\tau \simeq 1/\Gamma_t \simeq 5 \times 10^{-25}$ s
- typical hadronization time $\tau_{\rm had} \simeq 1/\Lambda_{\rm QCD} \simeq 2 \times 10^{-24}$ s
- $\tau < \tau_{\rm had} \Rightarrow$ top decays before it forms hadrons
- Schrödinger eq: $\left(\frac{\Delta}{m^2} - \frac{\alpha_s\, C_F}{r} + \delta V - (E + i\Gamma_t)\right) G(\vec{r}, \vec{r}\,', E) = \delta(\vec{r} - \vec{r}\,')$
- poles (bound states) become a bump (would-be bound state)
- position of bump $\Rightarrow$ determination of mass
- height and width of bump $\Rightarrow$ determination of Γ_t
- typical scale: $\mu \simeq 2\, m\, v \simeq 2 \left(m\sqrt{E^2 + \Gamma_t^2}\right)^{1/2} \gtrsim 30$ GeV $\Rightarrow$ perturbation theory

Top threshold scan at linear collider [Pineda, AS]

no resummation of $\log v$

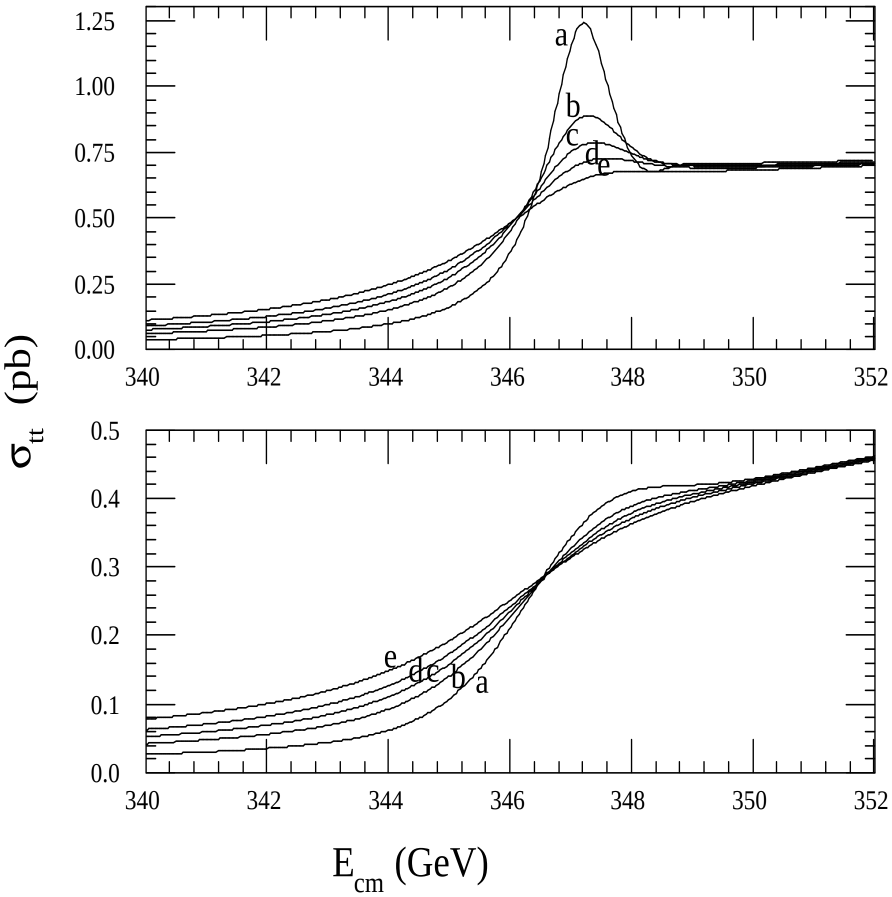


with resummation of $\log v$

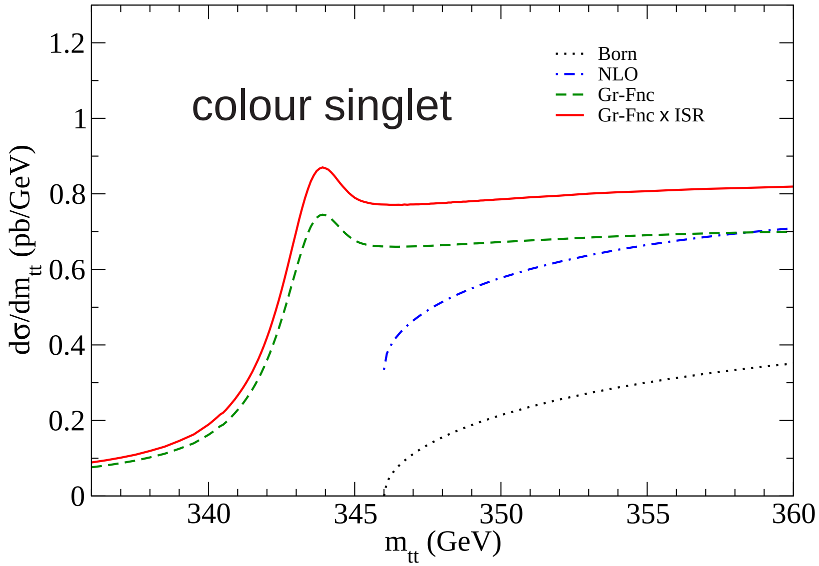


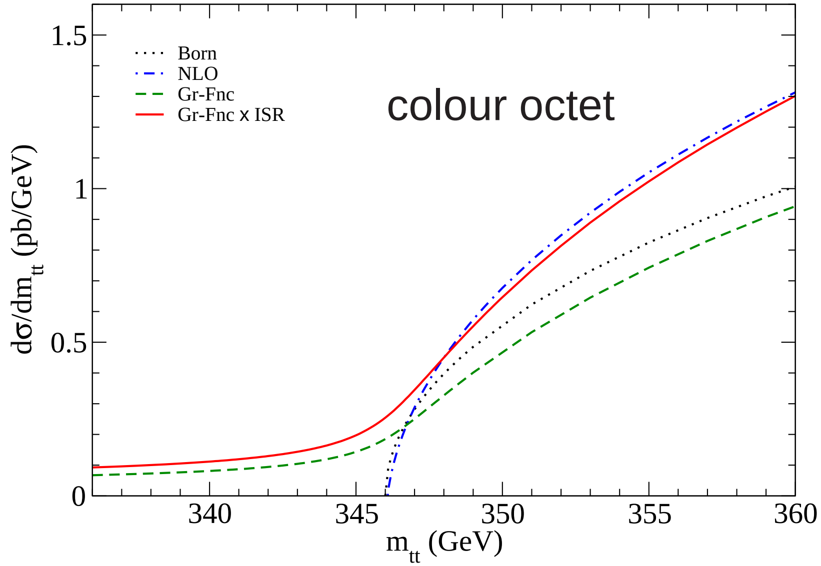
- normalization of cross section much more stable after resummation
- smaller scale dependence, smaller size of corrections
- potential to measure top mass to an accuracy of $\delta m_t \simeq 200$ MeV
- potential for a precise measurement of Γ_t and maybe even the Yukawa coupling

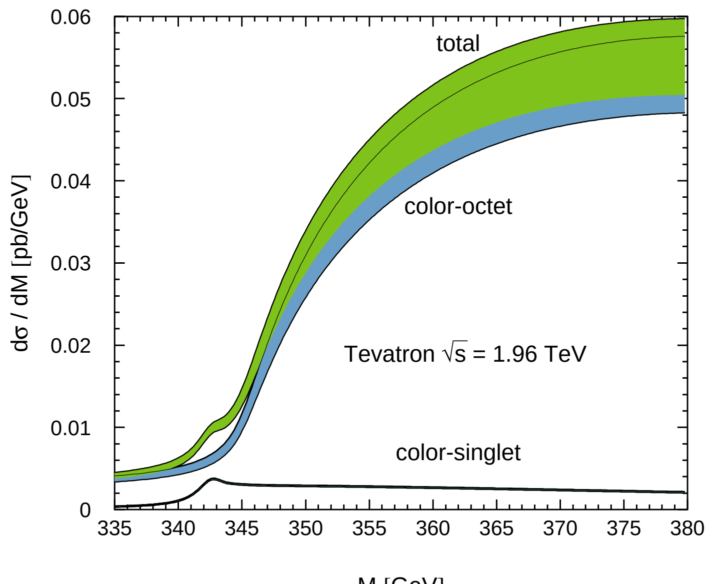
Top threshold scan at linear collider

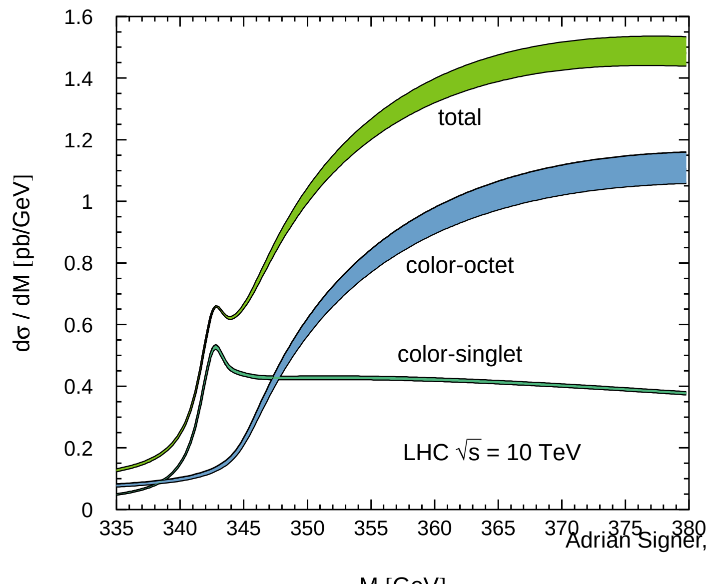


Top “threshold scan” at LHC [Hagiwara et.al.; Kiyo et.al.]









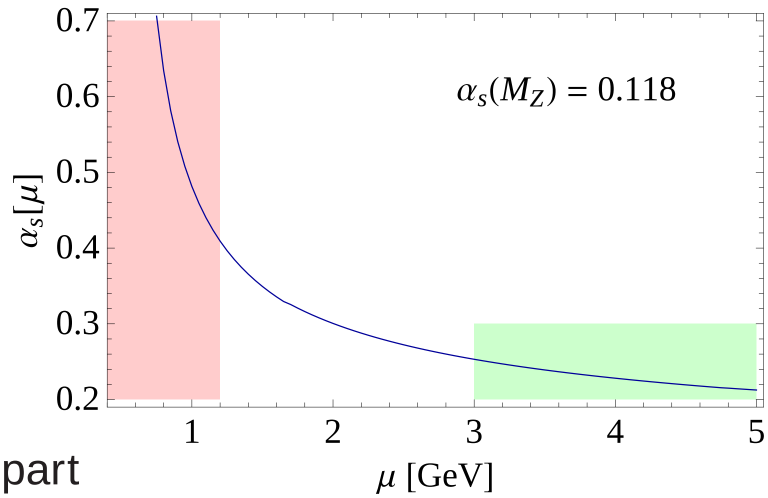
Extraction of bottom mass from $M_{\Upsilon(1S)}$

$M_{\Upsilon(1S)} = 2m_b + E \quad \Rightarrow \quad m_{b,\mathrm{PS}} = (4.58 \pm 0.04\,\text{(th)} \pm 0.07\,\text{(non-pert)})\ \mathrm{GeV}$

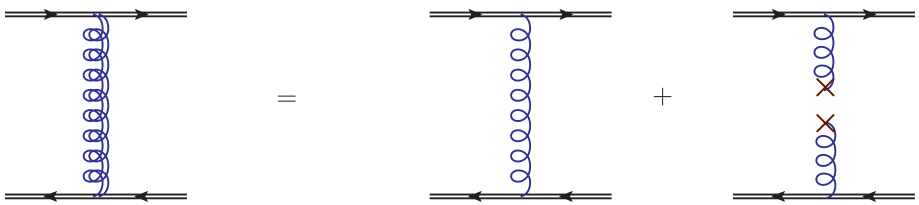
scales: $E_n \simeq -\dfrac{(C_F\,\alpha_s)^2 m}{4n^2} \quad \Rightarrow$

$\mu_s \sim p_n \simeq \dfrac{C_F\,\alpha_s\,m}{2\,n} \simeq$ 1.2 GeV for $n = 1$?!

non-perturbative effects !



full gluon propagator = pert. part $(1/q^2)$ + non-pert. part



parameterization of QCD at low energies through condensates

leading: $\alpha_s\,\langle G^a_{\mu\nu} G^a_{\mu\nu}\rangle \quad \Rightarrow \quad$ contribution $\delta M^{\mathrm{np}}_{\Upsilon(1S)} = \dfrac{624}{425}\pi\, m \dfrac{\langle G^a_{\mu\nu} G^a_{\mu\nu}\rangle}{(m C_F \alpha_s)^4} \simeq 70 - 90\ \mathrm{MeV}$

subleading: $g_s^3 f^{abc} \langle G^a_{\mu\nu} G^b_{\nu\lambda} G^c_{\lambda\mu}\rangle \quad \ldots$

to be continued