Nonrelativistic QED and QCD

Aneesh Manohar

based on earlier work:
Caswell and Lepage (NRQED)
Bodwin, Braaten, Lepage (NRQCD)

Brambilla, Pineda, Soto, Vairo (pNRQCD)
Beneke

Initial work with Michael Luke and Ira Rothstein, trying to understand how to explain some of Beneke's 2-loop computations using pNRQCD. with Iain Stewart and Andre Hoang. Carried to 3 loops in recent work by Hoang

General concepts of EFT and the reasons for using them are discussed in detail in the lectures by Beneke.

- In some cases, the fundamental theory is not known, but one can still compute using an effective field theory. e.g. Majorana neutrino masses in the standard model can be included using dimension 5 operators.
- If the fundamental theory is known (e.g. QCD), then in certain regimes, it is more convenient to use an EFT.
  - The relevant degrees of freedom might be non-perturbative (chiral perturbation theory)

- Converts a multiscale problem into several much simpler single scale problems: e.g. weak interactions using

$$\frac{4G_F}{\sqrt{2}}\left[C_1\;\bar c_{L\beta}\,\gamma^\mu\,b_L^\beta\;\bar d_{L\lambda}\,\gamma_\mu\,u_L^\lambda + C_2\;\bar c_{L\beta}\,\gamma^\mu\,b_L^\lambda\;\bar d_{L\lambda}\,\gamma_\mu\,u_L^\beta\right]$$

- New emergent symmetries: heavy quark spin-flavor symmetry in HQET.
- Can sum logs of ratios of scales, which can be ~ 1, using the renormalization group:

$$\left[\alpha_s \ln\left(\frac{M_W}{m_b}\right)\right]^n, \qquad \left[\alpha_s \ln\left(\frac{m_b}{\Lambda_{\rm QCD}}\right)\right]^n,$$

and compute operator mixing.

Power Counting

EFT have a systematic expansion in some parameter.

Chiral perturbation theory is an expansion in $p^2 \sim m_\pi^2$, so the expansion parameter is $\lambda \sim p^2/\Lambda_\chi^2$.

One heavy particle: HQET is an expansion in powers of $1/m_b$, so L has an expansion in powers of ∂/m_b. In HQET power counting,

$$iD^0 \sim E \sim \mathcal{O}\left(1\right), \qquad \frac{(i\mathbf{D})^2}{2m_b} \sim \frac{\mathbf{p}^2}{2m_b} \sim \mathcal{O}\left(\frac{1}{m_b}\right)$$

Notation: $p^2 = (p^0)^2 - \mathbf{p^2}$, $p_\perp^2 = -\mathbf{p}^2$, $\partial = -\nabla$

HQET Power Counting

The leading order Lagrangian is

$$L_0 \;=\; \bar{b}_v \, iD^0 \, b_v$$

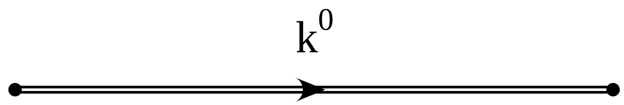
and

$$L_1 \;=\; -\bar{b}_v \, \frac{(i\mathbf{D})^2}{2m_b} \, b_v - a(\mu) \, g \, \bar{b}_v \, \frac{\sigma_{\mu\nu} G^{\mu\nu}}{4m_b} \, b_v$$

is treated as a correction.

L_0 contributes to the B meson mass at order $\Lambda_{\rm QCD}$, and L_1 gives corrections of order $\Lambda_{\rm QCD}^2/m_b$.

The heavy quark propagator is



$$\frac{i}{k^0 + i0^+} \to \theta(x^0)\,\delta(\mathbf{x})$$

- The quark propagates forwards in time.
- The quark is static, i.e. it does not move in $\mathbf{x}$.
- No antiquarks in b_v

NRQCD

Nonrelativistic Bound States of Two Heavy Particles:

$p\,e^-$	Hydrogen (H)	NRQED
$e^+\,e^-$	Positronium (Ps)	NRQED
$\mu^+\,e^-$	Muonium	NRQED
$b\bar{b}$, $c\bar{c}$, $b\bar{c}$	Υ, J/Ψ, B_c	NRQCD
$t\bar{t}$	$e^+e^- \to t\bar{t}$	NRQCD
NN	Deuteron	Few nucleon EFT
$\pi^+\pi^-$	Pionic bound states	

Study the spectroscopy, decays and production, and include radiative corrections, relativistic corrections, and nonperturbative effects in a systematic way.

In QED, the expansion parameter is $\alpha \sim 1/137 \ll 1$. Nevertheless, one cannot always use perturbation theory in α.

Hydrogen Atom: One needs to solve the Schrödinger equation with the potential

$$V = -\frac{\alpha}{r}$$

The Schrödinger equation sums up multiple iterations of the Coulomb potential. The energies can be expressed in a series in α [but the wavefunctions cannot].

Hydrogen

The Hydrogen atom (Bohr Formula)

$$H = \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r}, \qquad \frac{1}{m_r} = \frac{1}{m_p} + \frac{1}{m_e}$$

$$E_n = -\frac{m_r \alpha^2}{2n^2}, \qquad a_0 = \frac{1}{m_r \alpha}$$

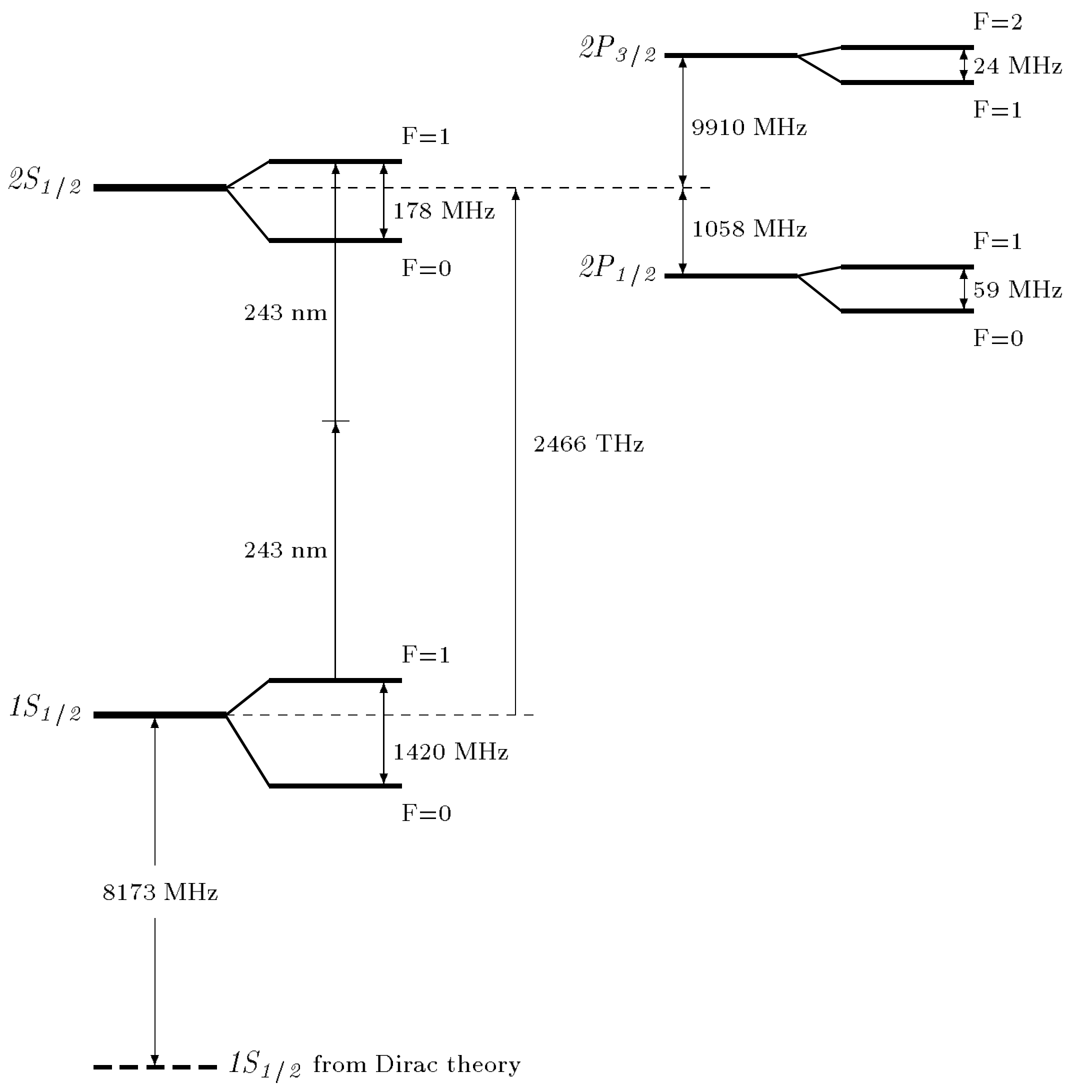
So

$$T \sim m\alpha^2, \qquad V \sim m\alpha^2, \qquad p \sim m\alpha, \qquad r \sim \frac{1}{m\alpha}$$

Hydrogen Atom

n-dependence: α^2
fine structure: α^4
Lamb shift: $\alpha^5 \ln \alpha$
hyperfine structure:
$\alpha^4 \; m_e/m_p$

$m_e \sim 7.5 \times 10^{14}$ MHz
$m_e\alpha^2 \sim 4 \times 10^{10}$ MHz



Nonrelativistic Power Counting

Nonrelativistic particles with velocity $v \ll 1$

$$E \sim mv^2, \qquad p \sim mv$$

If the the heavy quark interacts with light degrees of freedom, such as light quarks and gluons, then

$$\Delta E \sim \Lambda_{\rm QCD}, \qquad \Delta p \sim \Lambda_{\rm QCD}$$

and so $\Delta v = 0$ in the $m \to \infty$ limit.

But if there are two heavy particles, one can have $\Delta v \neq 0$ interactions in heavy–heavy interactions. The particles can orbit each other, and so can change their position $\mathbf{x}$.

Scales

Scale	Power Counting	
m_1, m_2, m_r	m	hard scale
p, $1/r$, a_0^{-1}	$mv \sim m\alpha$	soft scale
E	$mv^2 \sim m\alpha^2$	ultrasoft scale

The expansion parameter is v, and the RG will sum powers of $\ln v = \ln \alpha$.

In NRQCD, $v \sim \alpha_s$, and we also have $\Lambda_{\rm QCD}$. We will assume that $mv^2 \geq \Lambda_{\rm QCD}$, so that one has a Coulomb system with nonperturbative corrections.

True for $t\bar{t}$ near threshold, and for the Υ. Not a good approximation for J/Ψ.

Goal for the Effective Theory

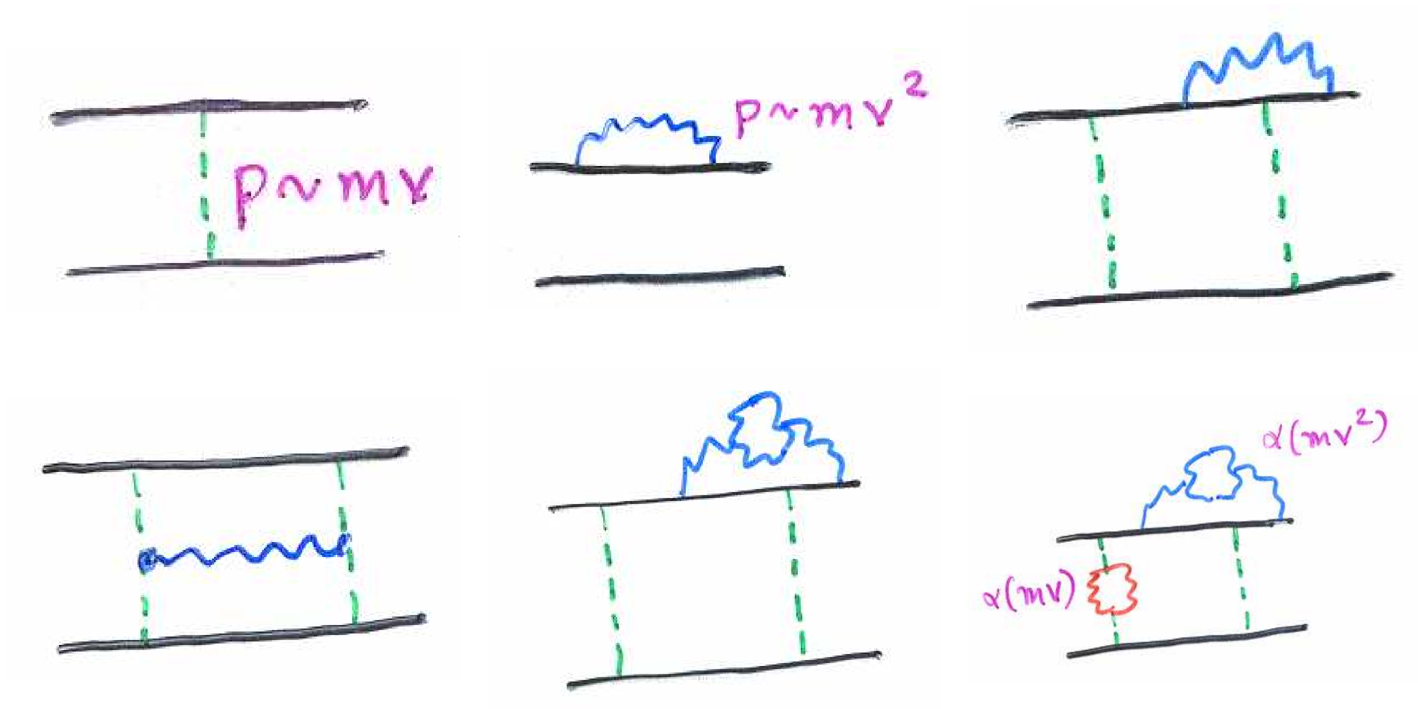
- Have a systematic expansion in some small parameter (v or α)
- Separate scales consistently
- Sum large logarithms (from ratios of scales) using the renormalization group

$$\ln \frac{p}{m}, \qquad \frac{1}{2} \ln \frac{E}{m}, \qquad \ln \frac{E}{p} \to \ln v \to \ln \alpha$$

- Determine scale for α_s:

$$\alpha_s(m), \qquad \alpha_s(mv), \qquad \alpha_s(mv^2)$$

What is the problem?



Graphs involve $\alpha_s(mv)$ and $\alpha_s(mv^2)$ at the same time.
Usually, one has $\alpha_s(\mu)$.

In the effective Lagrangian,

$$iD^0 = E \sim mv^2, \qquad \frac{(i\mathbf{D})^2}{2m} = \frac{\mathbf{p}^2}{2m} \sim mv^2$$

so both terms are the same order in the power counting.

The Coulomb interaction

$$V = -\frac{\alpha}{r} \sim \alpha\, mv$$

is also leading order if $v \sim \alpha$. So need to solve nonperturbatively in the Coulomb interaction, i.e. solve the Schrödinger equation.

Separate fields for the quark and antiquark, since the the theory is nonrelativistic.

ψ annihilates quarks, color $\mathbf{3}$
χ annihilates antiquarks, color $\bar{\mathbf{3}}$

$\psi^\dagger$ creates quarks
$\chi^\dagger$ creates antiquarks

The full relativistic field $Q \sim \psi + \chi^\dagger$

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi + ig\, T^a A^a_\mu\, \psi \\ D_\mu \chi &= \partial_\mu \chi + ig\, \bar{T}^a A^a_\mu\, \chi, \qquad \bar{T}^a = -(T^a)^t \end{aligned}$$

Note that Bodwin, Braaten, Lepage use $\chi \leftrightarrow \chi^\dagger$.

Quarks have $E \sim mv^2$, $p \sim mv$, and couple to massless gluons with $E \sim p$.

To have an EFT with manifest power counting, one can introduce two gluon fields, A_u and A_s with $E \sim p \sim mv^2$, and $E \sim p \sim mv$.

Break up the momentum into mv and mv^2 parts,

$$q \;=\; p + k$$

where p is order mv and k is order mv^2, similar to the

$$q \;=\; mv + k$$

breakup in HQET.

Since $E_n \sim m\alpha^2$, transitions between states emit photons of energy $m\alpha^2$, so emitted radiation can only be usoft.

Soft gauge bosons contribute to the quark potential.

So $p \sim mv$ becomes a label on the field, just as v is a label in HQET.

$$\psi_{\mathbf{p}}(x),\ \ \chi_{\mathbf{p}}(x) \longrightarrow (k^0, \mathbf{p}+\mathbf{k}),$$
$$A^\mu_{s,p}(x) \longrightarrow (p^0+k^0, \mathbf{p}+\mathbf{k})$$
$$A^\mu_{us}(x) \longrightarrow (k^0, \mathbf{k})$$

where the field has the x dependence $\sim e^{-ik\cdot x}$.

Convenient to introduce soft quarks,

$$\psi_{s,p\mathbf{p}}(x),\ \ \chi_{s,p}(x) \longrightarrow (p^0+k^0, \mathbf{p}+\mathbf{k}),$$

which are off-shell (auxiliarly fields).

RPI

A change

$$p \to p - l, \qquad k \to k + l$$

leaves $q = p + k$ invariant. l must be ultrasoft to maintain the mv^2 power counting for k.

In terms of fields,

$$\begin{aligned} \psi_{\mathbf{p}}(x) &\to e^{il\cdot x}\ \psi_{\mathbf{p}-\mathbf{l}}(x) \\ A^{\mu}_{s,p}(x) &\to e^{il\cdot x}\ A^{\mu}_{s,p-l}(x) \end{aligned}$$

For quarks, $l^0 = 0$.

RPI

The NRQCD Lagrangian should be RPI invariant. So derivatives must enter in the form

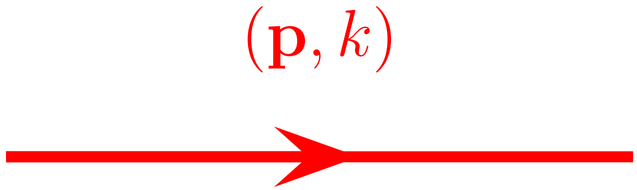
$$p^\mu + iD^\mu, \qquad \mathbf{p} + i\mathbf{D}$$

The quark kinetic energy is

$$S = \sum_{\mathbf{p}} \int \mathrm{d}^4x \ \psi^\dagger_{\mathbf{p}}(x) \left[iD^0 - \frac{(\mathbf{p} + i\mathbf{D})^2}{2m} \right] \psi_{\mathbf{p}}(x)$$

For power counting: $\mathbf{p} \sim mv$, $i\mathbf{D} \sim mv^2$, so the leading order Lagrangian is

$$S = \sum_{\mathbf{p}} \int \mathrm{d}^4x \ \psi^\dagger_{\mathbf{p}}(x) \left[iD^0 - \frac{\mathbf{p}^2}{2m} \right] \psi_{\mathbf{p}}(x) + \ldots$$



The $\psi_{\mathbf{p}}$ propagator is:

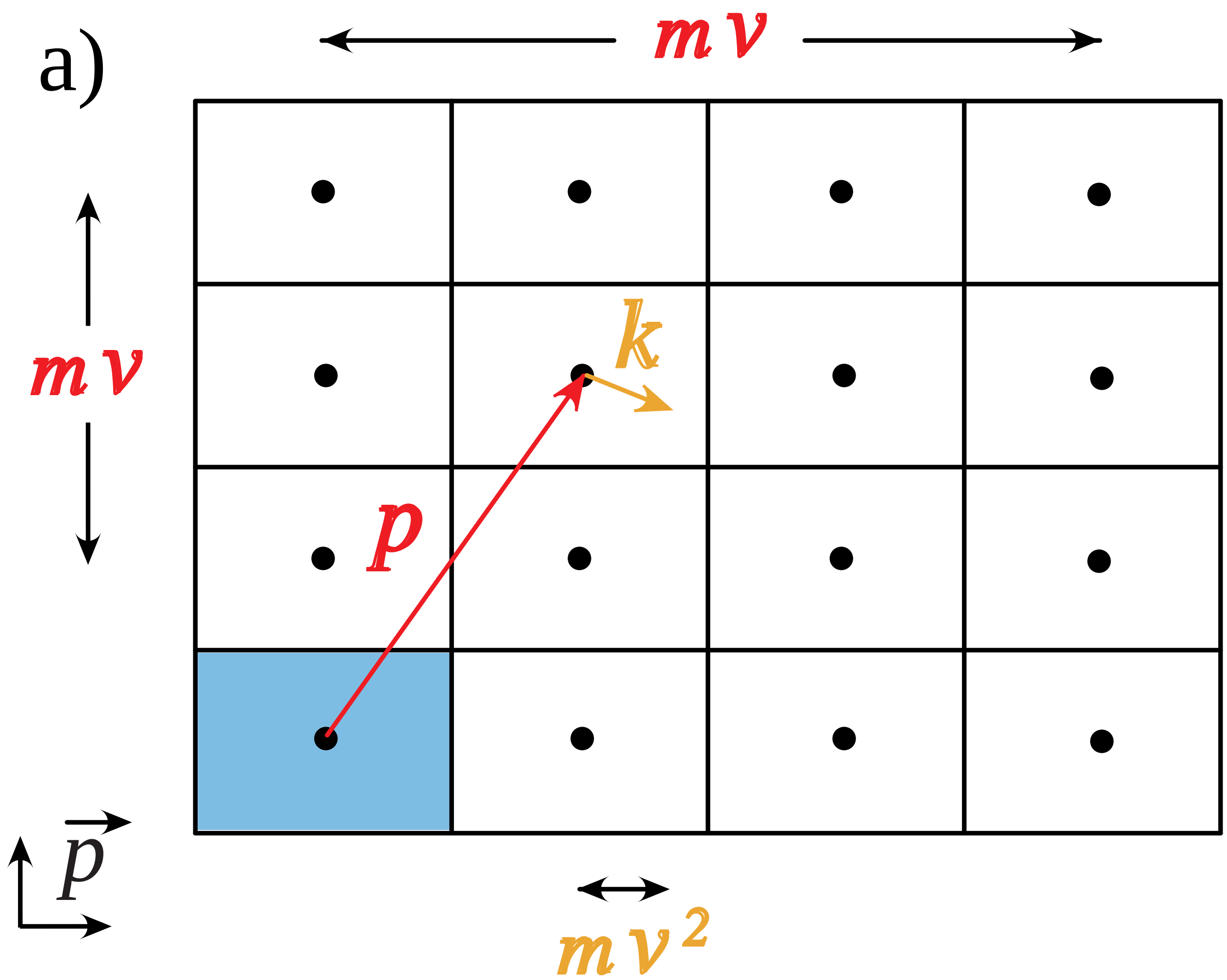
$$\frac{i}{k^0 - \frac{\mathbf{p}^2}{2m} + i0^+}$$

$i\mathbf{D}$ terms should be treated as perturbations.

$$-\psi^\dagger_{\mathbf{p}}(x)\frac{i\mathbf{p}\cdot\mathbf{D}}{m}\psi_{\mathbf{p}}(x)$$

The usual $p \cdot A$ interaction. But also comes with a $p \cdot k$ interaction.

a)



In loop graphs, one has

$$\sum_p \int \mathrm{d}k \to \int \mathrm{d}p$$

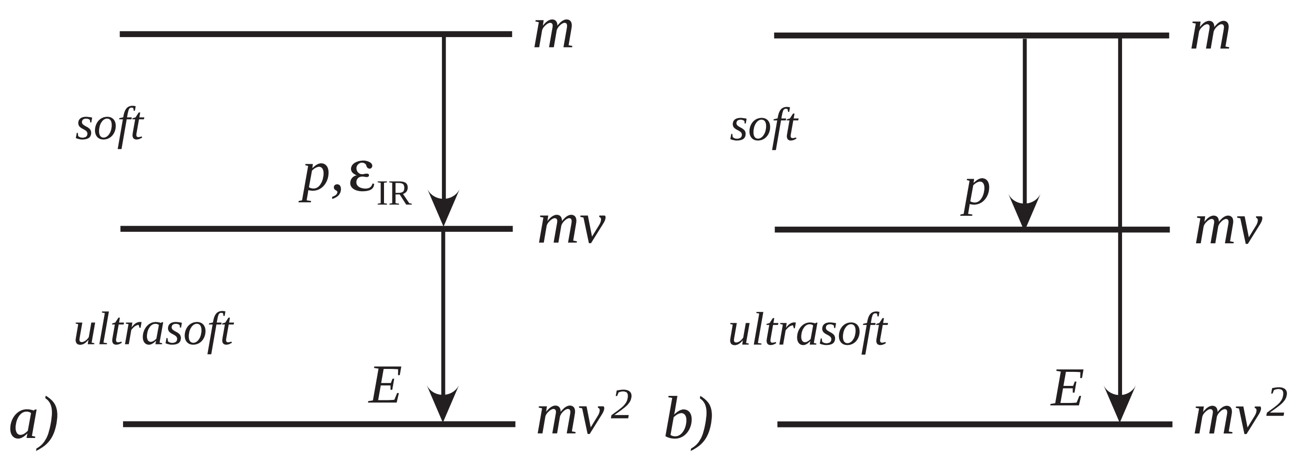
Or you can think of p as a continuous variable, and

$$\int \frac{\mathrm{d}p \ \mathrm{d}k}{[\mathrm{RPI}]} \to \int \mathrm{d}p$$

Regions

Beneke and Smirnov using the method of regions (threshold expansion) have shown that there are hard, potential, soft and ultrasoft regions.

Region	E	p	Field	Scaling
Hard	m	m	Integrated out	
Potential	mv^2	mv	ψ, χ	$v^{3/2}$
Soft	mv	mv	A_s^μ	v
			ψ_s, χ_s, q_s	$v^{3/2}$
Ultrasoft	mv^2	mv^2	A_{us}^μ	v^2
			q_{us}	v^3

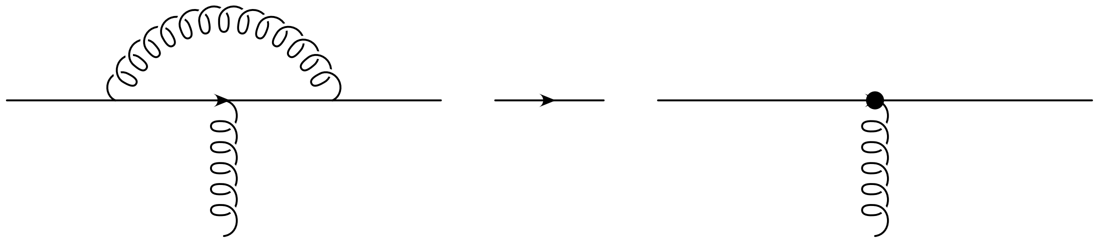


l.h.s. usual picture. IR divergences in the hard region match with UV divergences in the soft, and IR in the soft match with UV in the usoft.

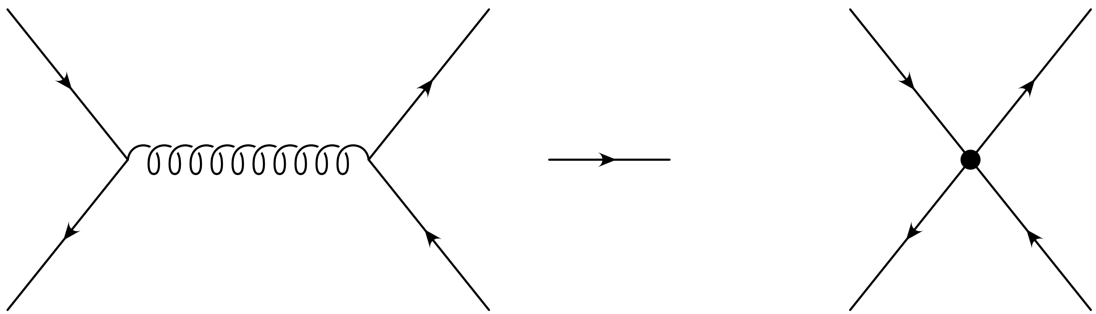
But not all IR in the hard match onto the soft, some match onto the usoft sector. Need both modes in theory at the same time. correlated scales

Hard Contributions

vertex corrections:



annihilation: virtual boson has $E \sim 2m$

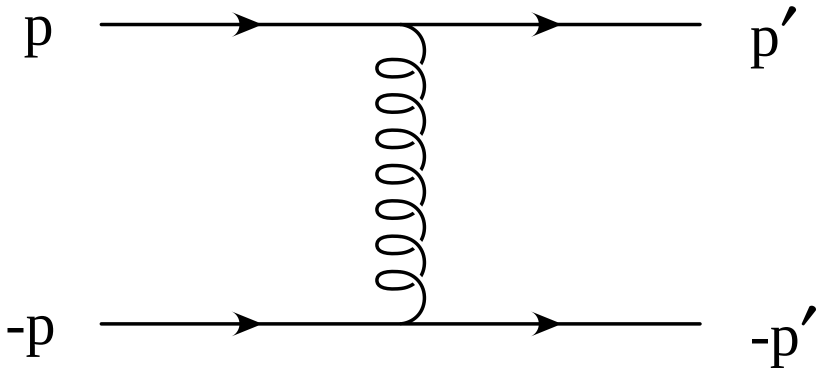


HQET-like corrections to gluon couplings, as well as four-quark operators $\sim \psi^\dagger \chi^\dagger \psi \chi$.

Compute these corrections as for HQET: Compute the graph in the full theory, and expand in powers of E/m and p/m

Potential

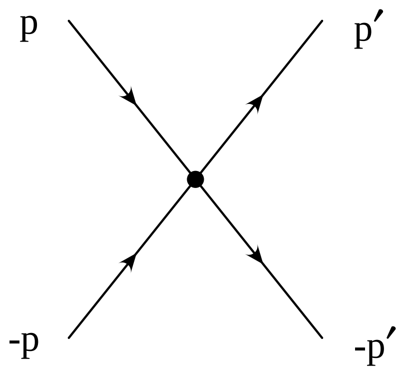
Look at quark-antiquark scattering:



$$k^0 \ll \mathbf{k} = \mathbf{p} - \mathbf{p}'$$

$$-i\frac{\mathcal{V}_c}{(\mathbf{p}-\mathbf{p}')^2}\; T^A \otimes \bar{T}^A, \qquad \mathcal{V}_c = 4\pi\alpha_s$$

Potential

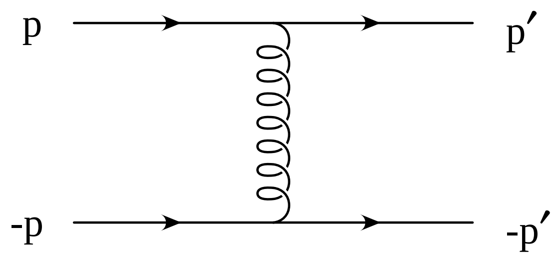


$$V_c(\mathbf{p},\mathbf{p}') = \frac{\mathcal{V}_c}{(\mathbf{p}-\mathbf{p}')^2}\; T^A \otimes \bar{T}^A,$$

$T^A \otimes \bar{T}^A$ is $-C_F = -4/3$ in the singlet channel, and $C_A/2 - C_F = 1/6$ in the octet channel.

$$L \;=\; -\sum_{\mathbf{p},\mathbf{p}'} V(\mathbf{p},\mathbf{p}')\; \psi^\dagger_{\mathbf{p}'} T^A \psi_{\mathbf{p}}\, \chi^\dagger_{-\mathbf{p}'} \bar{T}^A \chi_{-\mathbf{p}}$$

Exercise: Compute



and expand in powers of $1/m$ to get the potential at tree level.

$$\frac{[\bar{u}(\mathbf{p}')\,\gamma^{\mu}\,u(\mathbf{p})]\,[\bar{u}(-\mathbf{p}')\,\gamma_{\mu}\,u(-\mathbf{p})]}{2E_p 2E_{p'}(\mathbf{p}-\mathbf{p}')^2}$$

Compute on-shell in Feynman and Coulomb gauge.

$$
\begin{aligned}
V^{(-1)} &= \frac{U_c}{\mathbf{k}^2}, \\
V^{(0)} &= \frac{U_k}{|\mathbf{k}|}\,, \\
V^{(1)} &= U_2 + U_s\,\mathbf{S^2} + \frac{U_r(\mathbf{p}^2+\mathbf{p}'^2)}{2\mathbf{k}^2} - \frac{i\mathbf{U}_\Lambda\cdot(\mathbf{p}'\times\mathbf{p})}{\mathbf{k}^2} \\
&\qquad + U_t\left(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \frac{3\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2}{\mathbf{k}^2}\right),
\end{aligned}
$$

Allowed terms are functions of $\mathbf{p}, \mathbf{p}'$. Seems as though there are lots of them.

Write them as a function of $\mathbf{k} = \mathbf{p}' - \mathbf{p}$ and $\mathbf{P} = \mathbf{p}' + \mathbf{p}$.

A quantum mechanics potential $V(r)$ becomes $\tilde{V}(\mathbf{k})$, and $\{\hat{\mathbf{p}}, V(r)\}$ becomes $\mathbf{P}\tilde{V}(\mathbf{k})$, etc.

The only singularity in the Coulomb problem is from the $1/r$ behavior of the potential at long distances; there is no singularity for backscattering $\mathbf{P} = 0$.

There can be at most two powers of $\mathbf{k}$ and no powers of $\mathbf{P}$ in the denominator.

$|\mathbf{k}|$ does not occur at tree-level.

$\mathbf{S}$ comes with $1/m$.
Particles of mass $m_{1,2}$ and charge $-e$, Ze

$$
\begin{aligned}
U_c &= -4\pi Z\alpha \\
U_2 &= \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)^2 \\
U_s &= \frac{4\pi Z\alpha}{3m_1m_2}+\frac{\pi\alpha}{m_e^2} \\
U_r &= -\frac{4\pi Z\alpha}{m_1m_2} \\
U_k &= \frac{\pi^2Z^2m_r\alpha^2}{m_1m_2}
\end{aligned}
$$

There can be terms which have the form (V_Δ)

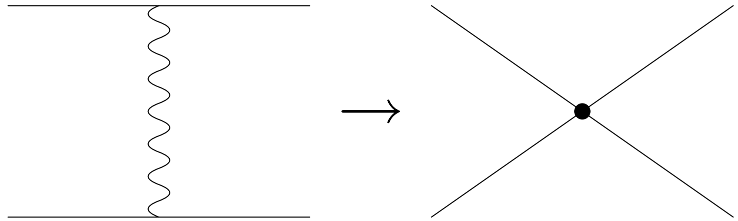
$$\frac{\mathbf{k}^i\mathbf{k}^j}{\mathbf{k}^4}\mathbf{P}^i\mathbf{P}^j = \frac{(\mathbf{p}^2 - \mathbf{p}'^2)^2}{\mathbf{k}^4}$$

in Coulomb gauge, or $(E - E')^2$ in Feynman gauge. These vanish on-shell.

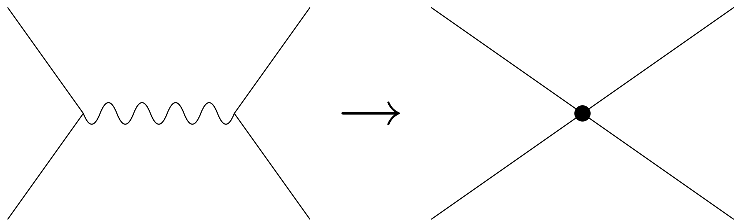
A field redefinition converts them to the $1/\left|\mathbf{k}\right|$ potential.

Exercise: Compute the $V_\Delta - V_c$ loop graph.

Annihilation



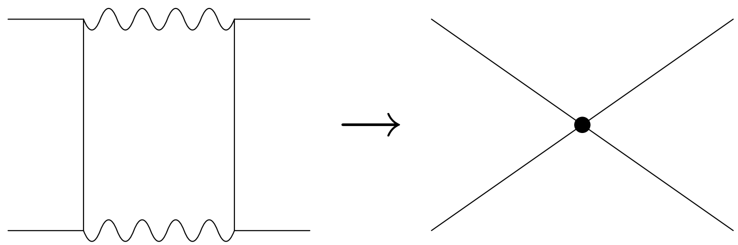
Only difference between Hydrogen and Positronium is annihilation contributions to the potentials, that first start at order $1/m^2$:



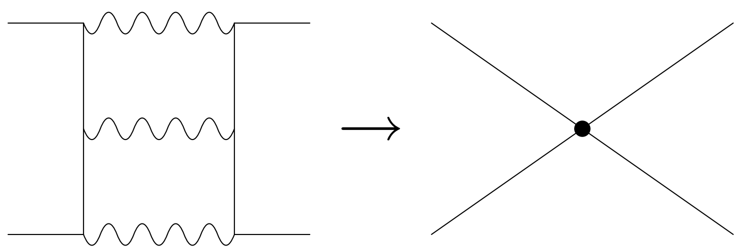
$$U_2 + U_s \mathbf{S}^2 = \frac{\pi\alpha\mathbf{S}^2}{m_e^2}$$

Widths

[Zero in Singlet vs Triplet]



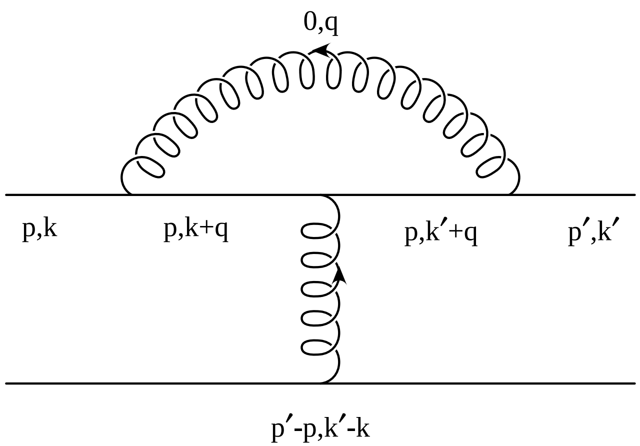
$$U_2 + U_s\mathbf{S}^2 \;\;=\;\; -i\frac{\pi\alpha^2}{m_e^2}\left(2-\mathbf{S}^2\right)$$



$$U_2 + U_s\mathbf{S}^2 \;\;=\;\; -i\frac{4\pi\alpha^3\left(\pi^2-9\right)}{9\pi m_e^2}\mathbf{S}^2$$

Loops

An ultrasoft gluon correction to Coulomb scattering:



$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 + i0^+} \frac{1}{k^0 + q^0 - \mathbf{p}^2/(2m) + i0^+} \frac{1}{k'^0 + q^0 - \mathbf{p}'^2/(2m) + i0^+}$$

$$\times V(\mathbf{p}', \mathbf{p})$$

Loops

k, q have been expanded out:

$V(\mathbf{p}', \mathbf{p})$ independent of the loop momentum.

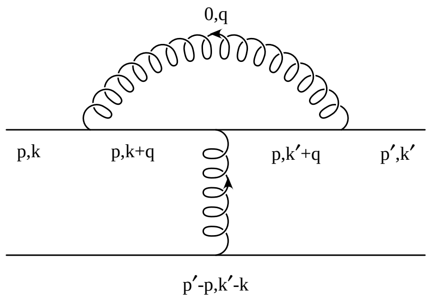
Quark propagator:

$$\frac{1}{k^0 + q^0 - (\mathbf{p}+\mathbf{k}+\mathbf{q})^2/(2m) + i0^+} \quad \rightarrow \quad \frac{1}{k^0 + q^0 - \mathbf{p}^2/(2m) + i0^+}$$

Integrals in dimensional regularization can be done by residues, using the poles in the denominator.

l.h.s: $q^0 \sim \mathbf{q} \sim m$ pole destroys the EFT expansion.

r.h.s does not.



Label conservation at each vertex, and usoft momentum conservation at each vertex from the $\int \mathrm{d}^4x$ in L.

This is the momentum space version of the multipole expansion:

$$A(x) \sim a_k^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \rightarrow a_k^\dagger(1 - i\mathbf{k}\cdot\mathbf{x} + \ldots)$$

This acts on wavefunctions with momentum $\mathbf{p}$

$$e^{-i\mathbf{k}\cdot\mathbf{x}}\ e^{i\mathbf{p}\cdot\mathbf{x}} = e^{i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}}$$

but

$$(1 - i\mathbf{k}\cdot\mathbf{x} + \ldots)\ e^{i\mathbf{p}\cdot\mathbf{x}}$$

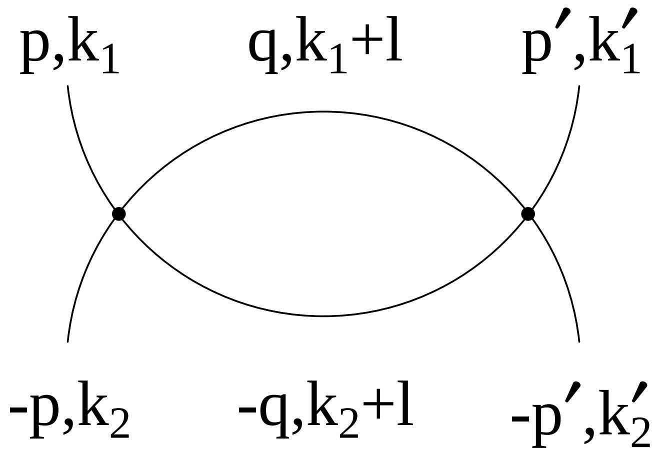
does not change the momentum to any finite order in the expansion.

Go back to the diagram, and assume external particles are on-shell, so that $k^0 = \mathbf{p}^2/2m$. Then

$$\frac{1}{k^0 + q^0 - \mathbf{p}^2/(2m) + i0^+} \quad \longrightarrow \quad \frac{1}{q^0 + i0^+}$$

and the corrections look like HQET.

Potential Loops



$$I \;=\; \sum_{\mathbf{q}} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} (-i)^2 (i)^2 \; V(\mathbf{p}', \mathbf{q}) \; V(\mathbf{q}, \mathbf{p})$$

$$\times \frac{1}{k_1^0 + l^0 - \mathbf{q}^2/(2m_1) + i0^+} \; \frac{1}{k_2^0 - l^0 - \mathbf{q}^2/(2m_1) + i0^+}$$

RPI invariant integration says that

$$\sum_{\mathbf{q}} \int \mathrm{d}^4 l \;\; \rightarrow \;\; \int \mathrm{d}l^0 \; \mathrm{d}^3\mathbf{q}$$

Do the l^0 integral by residues:

$$\begin{aligned}
& \int \frac{\mathrm{d}l^0}{2\pi} \frac{1}{k_1^0 + l^0 - \mathbf{q}^2/(2m_1) + i0^+} \; \frac{1}{k_2^0 - l^0 - \mathbf{q}^2/(2m_1) + i0^+} \\
= \;\; & i \; \frac{1}{k_1^0 + k_2^0 - \mathbf{q}^2/(2m_r) + i0^+} \\
= \;\; & i \; \frac{2m_r}{2m_r(E_1 + E_2) - \mathbf{q}^2 + i0^+}
\end{aligned}$$

$$I = i\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} V(\mathbf{p}',\mathbf{q}) \; \frac{2m_r}{2m_r(E_1+E_2)-\mathbf{q}^2+i0^+} \; V(\mathbf{q},\mathbf{p})$$

Since $E_t = E_1 + E_2$ is of order mv^2, the denominator structure does not violate the power counting.

Iteration of the potential with the Schrödinger Green's function.

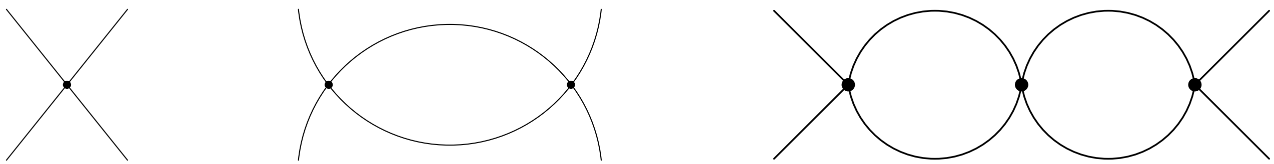
Note that there can be factors of m in the numerator.

The potential loops are sensitive to the energy via $\sqrt{2m_rE}$.

For the Coulomb potential,

$$V \sim \frac{\alpha}{\mathbf{q}^2}, \qquad I \sim \frac{m\alpha^2}{\mathbf{q}^3}$$

The one-loop graph is order α/v relative to the tree-level graph. Since $v \sim \alpha$, have to sum all the bubbles.



This is solving the Schrödinger equation in a Coulomb potential.

In dimensional regularization, introduce a scale parameter μ. Then loop graphs give $\ln p/\mu$.

Can get

$$\ln \frac{p}{\mu}, \qquad \ln \frac{E}{\mu}$$

In our case, one needs two scale parameters, μ_U and μ_S. μ_U multiplies ultrasoft interactions, and μ_S multiplies soft and potential interactions.

Better to think of a subtraction velocity ν rather than a subtraction scale μ. This makes sense, since the power counting parameter is a velocity, rather than a momentum scale.

$$\mu_S = m\nu, \qquad \mu_U = m\nu^2$$

Loop graphs in the effective theory give

$$\ln \frac{E}{\mu_U}, \qquad \ln \frac{\sqrt{2mE}}{\mu_S}, \qquad \ln \frac{\mathbf{q}}{\mu_S}$$

All logarithms are small if $\nu \sim v$. Checked to three loop order by Hoang.

A theory with correlated scales and correlated running.

Start at $\nu = 1$, where $\mu_S = \mu_U = m$. This is the matching scale between QCD and NRQCD, and the matching coefficient logarithms (hard graphs) are minimized.

Run to $\nu = v$. Logarithms of v are summed by the velocity renormalization group (VRG) equations.

At $\nu = v$, compute bound state matrix elements.. There are no large logarithms remaining.

All the large logarithms are summed by the VRG. These are

$$\ln\frac{m}{E},\qquad \ln\frac{m}{p},\qquad \ln\frac{m}{\sqrt{2mE}},\qquad \ln\frac{p}{E}$$

which are all of the form

$$\ln v \sim \ln\alpha$$

Power Counting

To get the v counting rules, need to assign a power counting to all the fields. Work in one time and $3-2\epsilon$ space dimensions.

$$\sum_{\mathbf{p}} \int \mathrm{d}k \;\to\; \int \mathrm{d}\mathbf{p}$$

$$\sum_{p_\mu} \int \mathrm{d}k \;\to\; \int \mathrm{d}p$$

$$\left[\sum_{\mathbf{p}}\right][k]^{3-2\epsilon} \quad = \quad [\mathbf{p}]^{3-2\epsilon}$$

$$[k] = mv^2, \qquad [\mathbf{p}] = mv$$

$$\left[\sum_{\mathbf{p}}\right] = v^{-(3-2\epsilon)}, \qquad \left[\sum_{p_\mu}\right] = v^{-(4-2\epsilon)}$$

$$\delta_{\mathbf{p}} = v^{(3-2\epsilon)}, \qquad \delta_{p_\mu} = v^{(4-2\epsilon)}$$

$$
\begin{aligned}
S &= \sum_{\mathbf{p}} \int \mathrm{d}^D x \ \psi^\dagger_{\mathbf{p}}(x) \left[iD^0 - \frac{\mathbf{p}^2}{2m} \right] \psi_{\mathbf{p}}(x) \\
&= \sum_{\mathbf{p},\mathbf{p}'} \delta_{\mathbf{p},\mathbf{p}'} \int \mathrm{d}^D x \ \psi^\dagger_{\mathbf{p}}(x) \left[iD^0 - \frac{\mathbf{p}^2}{2m} \right] \psi_{\mathbf{p}'}(x)
\end{aligned}
$$

One δ_p at each vertex, and a $\sum_p$ for each field.

$$
1 = v^{(3-2\epsilon)} (mv^2)^{-(4-2\epsilon)} (mv^2) \left[\sum_{\mathbf{p}} \psi_{\mathbf{p}} \right]^2
$$

$$
\left[\sum_{\mathbf{p}} \psi_{\mathbf{p}} \right] = (mv)^{3/2-\epsilon}
$$

Ultrasoft gluons have $E \sim p \sim mv^2$.

$$
\begin{aligned}
S &= -\frac{1}{4}\int \mathrm{d}^4x \; G^{\mu\nu}G_{\mu\nu} \\
1 &= (mv^2)^{-(4-2\epsilon)}(mv^2)^2\,[A_{us}]^2 \\
[A_{us}] &= (mv^2)^{1-\epsilon}
\end{aligned}
$$

Soft gluons have $E \sim p \sim mv$, and so

$$
\left[\sum_{p_\mu} A_{s,p}\right] = (mv)^{1-\epsilon}
$$

Covariant derivative:

$$iD = i\partial - g_U A_U \rightarrow i\partial - \mu_U^{\epsilon}\, g_U A_U$$

$$[\mu_U] = mv^2$$

Potential:

$$\begin{aligned} S &= \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} \int \mathrm{d}^D x\ \delta_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} \psi^\dagger_{\mathbf{p}_1} \psi_{\mathbf{p}_2} \chi^\dagger_{\mathbf{p}_3} \chi_{\mathbf{p}_4}\ V(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \\ &= (mv^2)^{-(4-2\epsilon)} v^{3-2\epsilon} \left[\sum_{\mathbf{p}} \psi_{\mathbf{p}}\right]^4 [V] \end{aligned}$$

$$= \quad (mv)^{-2\epsilon}(m^2 v)\,[V]$$

$[V]$ is some integer dimension quantity, so the potential terms must be multiplied by $\mu_S^{2\epsilon}$,

$$[\mu_S] = mv$$

and then a potential is of order

$$(m^2 v)\,[V]$$

The Coulomb potential is of order

$$V_c = -\frac{4\pi\alpha}{\left(\mathbf{p}_1 - \mathbf{p}_2\right)^2}, \qquad [V_c] = (m^2 v) \times \frac{\alpha}{m^2 v^2} = \frac{\alpha}{v}$$

$$\frac{\alpha^2}{|\mathbf{k}|} \quad \rightarrow \quad (m^2 v) \times \frac{\alpha^2}{mv} = m\alpha^2$$

so the coefficient is of order $1/m$, and the potential is of order α^2.

$$\alpha \times (1, \mathbf{S}^2, \ldots) \quad \rightarrow \quad (m^2 v) \times \alpha = m^2 \alpha v$$

so the coefficient is of order $1/m^2$, and the potential is of order αv. The V_Δ potential gets converted to the V_k potential. No m power counting.

Both two orders in $v \sim \alpha$ beyond Coulomb.

Contact interactions (e.g. the annihilation graph)

$$\frac{\alpha}{m^2}\psi^\dagger\psi\chi^\dagger\chi, \qquad\qquad [S] = \alpha\, v$$

spin-orbit interactions, etc. are treated as perturbations.

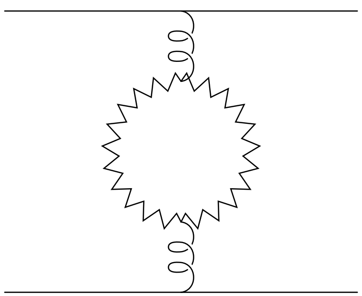
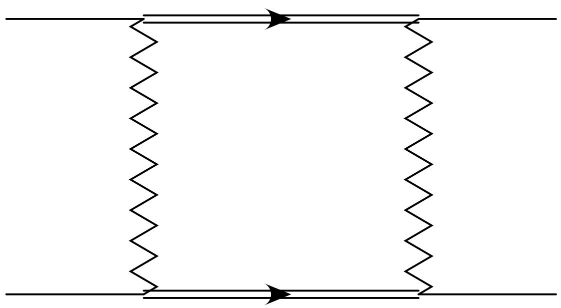
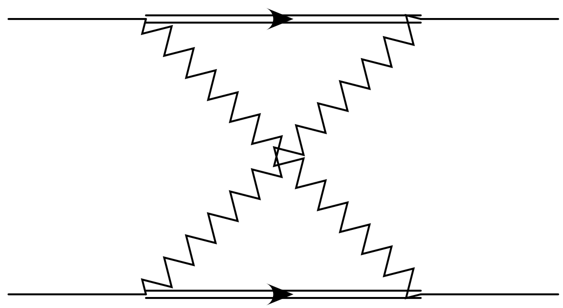
Soft Fields

Soft gluons have $E \sim p \sim mv$. There can be no single soft-gluon emission or absorption from the quarks, since energy cannot be conserved.

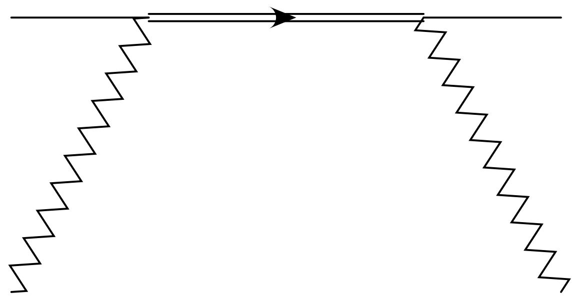
They can enter in pairs, as in Compton Scattering

$$g + q \rightarrow g + q$$

They enter in the running of α_s.

Soft Fields



The intermediate quarks are off-shell, since $E \sim mv$ and $p \sim mv$. It is convenient to introduce soft-quarks as auxiliary fields, $\psi_{s,q}$ with a four-momentum label, and propagator

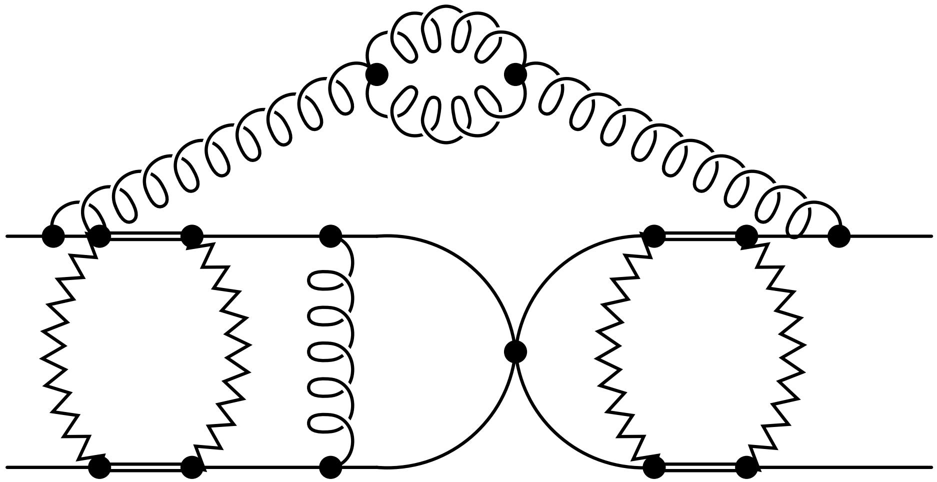
$$\frac{i}{q^0 + i0^+}$$

and scaling dimension $(mv)^{3/2-\epsilon}$.

Soft Fields

The soft sector is an auxiliary sector that generates a potential. It allows one to run the β-function terms.

Scaling for a Diagram

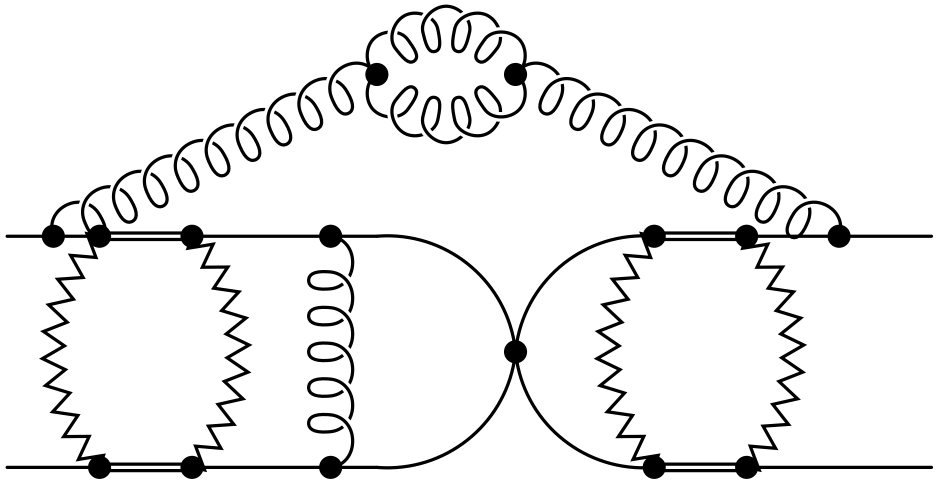


What is the order?

Usoft vertices (no δ), Potential vertices ($\delta_{\mathbf{p}}$) and soft vertices ($\delta_{\mathbf{p}_\mu}$): $V_U^{(k)}, V_P^{(k)}, V_S^{(k)}$

Usoft loops (no $\sum$), Potential loops ($\sum_{\mathbf{p}}$) and soft loops ($\sum_{\mathbf{p}_\mu}$): L_U, L_P, L_S

Total Diagram

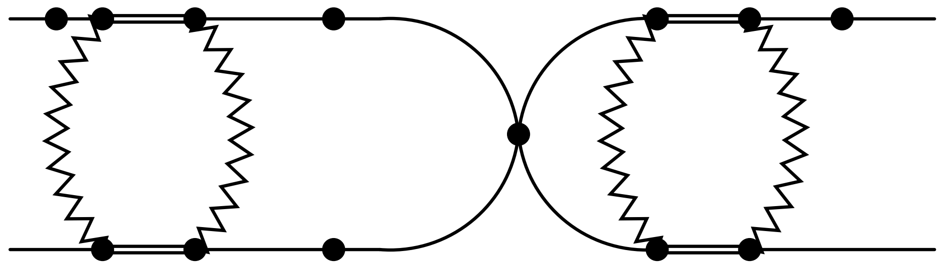


$$L_U + L_P + L_S = 7, \qquad \sum_k (V_U^{(k)} + V_P^{(k)} + V_S^{(k)}) = 15$$

$$\sum_k (V_U^{(k)} + V_P^{(k)} + V_S^{(k)}) - (I_U + I_P + I_S) + L_U + L_P + L_S = 1$$

V_U: only usoft, V_S: at least one soft, V_P: no soft, at least one potential,

Usoft Removed

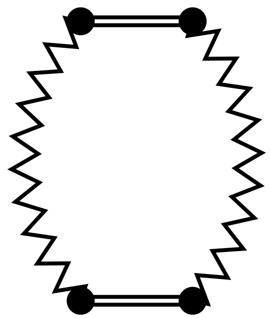


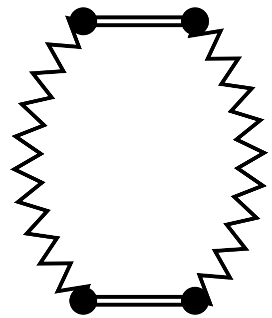
Delete all the usoft lines.

$$L_P + L_S = 4, \qquad \sum_k (V_P^{(k)} + V_S^{(k)}) = 13$$

$$\sum_k (V_P^{(k)} + V_S^{(k)}) - (I_P + I_S) + L_P + L_S = 1$$

Soft Removed





Delete all the potential lines

$$L_S = 2, \qquad \sum_k V_S^{(k)} = 8, \qquad N_S = 2$$

$$\sum_k V_S^{(k)} - I_S + L_S = N_S$$

$$L_U = 3, \qquad L_P = 2, \qquad \sum_k V_U^{(k)} = 2, \qquad \sum_k V_P^{(k)} = 5$$

Scaling for a Diagram

So net order in v is

$$\begin{aligned}\delta &= \sum_k k\left(V_U^{(k)} + V_P^{(k)} + V_S^{(k)}\right) \\ &\quad + \sum_k \left(8V_U^{(k)} + 5V_P^{(k)} + 4V_S^{(k)}\right) \\ &\quad -(4+4)I_U - (3+2)I_P - (2+2)I_S \\ &\quad +8L_U + 5L_P + 4L_S\end{aligned}$$

The measure is

$$\int \mathrm{d}^4x, \qquad \delta_{\mathbf{p}} \int \mathrm{d}^4x, \qquad \delta_{\mathbf{p}_\mu} \int \mathrm{d}^4x$$

Scaling for a Diagram

the internal lines remove two fields:

$$\langle A_U A_U \rangle \,, \qquad \left\langle \sum_{\mathbf{p}} \psi^\dagger_{\mathbf{p}} \sum_{\mathbf{p}} \psi_{\mathbf{p}} \right\rangle , \qquad \left\langle \sum_{p_\mu} A_{s,p} \sum_{p_\mu} A_{s,p} \right\rangle$$

and add

$$\frac{i}{p^2}, \qquad \frac{i}{k^0 - \mathbf{p}^2/(2m)}, \qquad \frac{i}{p_\mu^2}$$

Scaling for a Diagram

the loops add

$$\int \mathrm{d}^4 k, \qquad \sum_{\mathbf{p}} \int \mathrm{d}^4 k, \qquad \sum_{p_\mu} \int \mathrm{d}^4 k$$

Now get rid of I_U, I_P and I_S using the Euler constraints, and add back the measure for the potential interaction:

$$\delta = \sum_k k \left(V_U^{(k)} + V_P^{(k)} + V_S^{(k)} \right) - N_S$$

Scaling for a Diagram

Note that the Coulomb interaction is $1/v$, so it is a $V_P^{(-1)}$ vertex. But it comes with at least one power of α. So if we count power of α and v, there are no negative powers.

$-N_s$, but a soft component comes with at least α^2, so ≥ 1.

Can think of the soft component as equivalent to the V_P it generates.

Lamb Shift in QED

A.M., I. Stewart, PRL 85 (2000) 2248

			order	E
$V^{(-1)}$	$\frac{\alpha}{\mathbf{k}^2}$	$\frac{\alpha}{v}$	1	α^2
$V^{(0)}$	$\frac{\alpha^2}{m\lvert\mathbf{k}\rvert}$	α^2	α^2	α^4
$V^{(1)}$	$\frac{\alpha}{m^2}, \quad \frac{\alpha\mathbf{S}^2}{m^2}$	αv	α^2	α^4
$V^{(2)}$	$\frac{\alpha^2\lvert\mathbf{k}\rvert}{m^3}$	$\alpha^2 v^2$	α^4	α^6
$V^{(3)}$	$\frac{\alpha\mathbf{k}^2}{m^4}$	αv^3	α^4	α^6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

1. Sum Coulomb potential to all orders
2. $V^{(0)}V^{(0)}$, $V^{(0)}V^{(1)}$, $V^{(1)}V^{(1)} \sim \alpha^4 \to \alpha^6$ in E

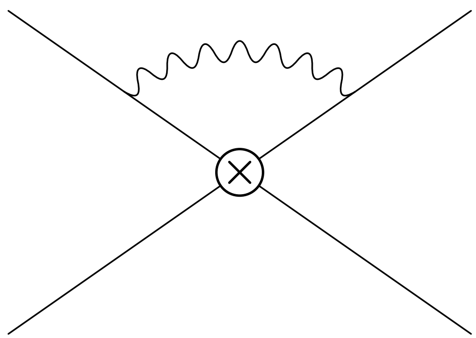
The ultrasoft interaction

$$D = \psi_{\mathbf{p}}^{\dagger} \frac{\mathbf{p} \cdot A_{us}}{m} \psi_{\mathbf{p}}$$

is of order $ev \to \alpha v^2$ at second order.
To order α^5 in energy, need only

$$\left\langle V^{(1)} \right\rangle + \langle T \{ V_c D D \} \rangle$$

and the corrections are of order α^6.



1. Match the coefficients at the scale m
2. Run them to $\nu = 1$
3. Compute Matrix elements

Concentrate here on the $\ln \alpha$ terms from the running.

Define LO and NLO anomalous dimensions relative to the leading term.

$\gamma_{\mathrm{LO}}, \gamma_{\mathrm{NLO}}$ are order α^2, α^3 for $V^{(1)}$ which is $\mathcal{O}(\alpha)$

$\gamma_{\mathrm{LO}}, \gamma_{\mathrm{NLO}}$ are order α^3, α^4 for $V^{(0)}$ which is $\mathcal{O}(\alpha^2)$

Since V's are of different orders, LO/NLO not related to the number of loops.

$$
\begin{aligned}
\gamma_{\mathrm{LO}} &: \quad \alpha^4 \quad\ \left(1+\alpha\ln\alpha+\alpha^2\ln^2\alpha+\alpha^3\ln^3\alpha+\ldots\right) \\
\gamma_{\mathrm{NLO}} &: \quad \alpha^4\alpha \ \left(1+\alpha\ln\alpha+\alpha^2\ln^2\alpha+\alpha^3\ln^3\alpha+\ldots\right) \\
\gamma_{\mathrm{NNLO}} &: \quad \alpha^4\alpha^2 \left(1+\alpha\ln\alpha+\alpha^2\ln^2\alpha+\alpha^3\ln^3\alpha+\ldots\right)
\end{aligned}
$$

γ_{NNLO} the same order as neglected terms.

So one can compute

$$
\begin{array}{cccc}
\alpha^5\ln\alpha & \alpha^6\ln^2\alpha & \alpha^7\ln^3\alpha & \ldots \\
\alpha^6\ln\alpha & \alpha^7\ln^2\alpha & \alpha^8\ln^3\alpha & \ldots
\end{array}
$$

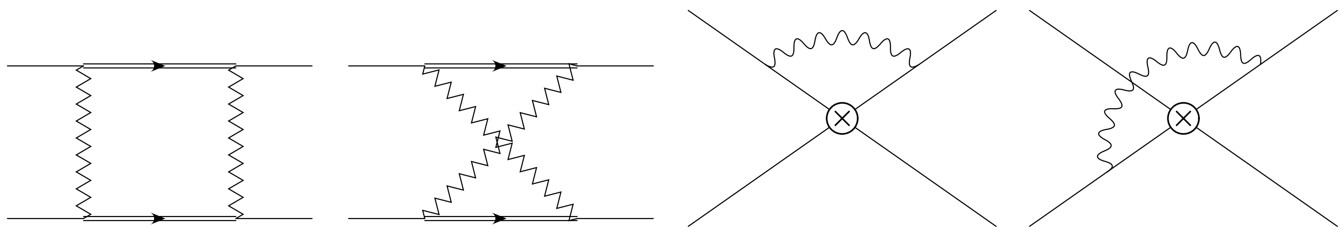
using γ_{LO}, γ_{NLO} for $V^{(0,1)}$.

$$V^{(-1)} = \frac{U_c}{\mathbf{k}^2} \qquad V^{(0)} = \frac{U_k}{|\mathbf{k}|}\,,$$

$$V^{(1)} = U_2 + U_s\,\mathbf{S^2} + \frac{U_r(\mathbf{p}^2 + \mathbf{p}'^2)}{2\mathbf{k}^2} - \frac{i\mathbf{U}_\Lambda\cdot(\mathbf{p}'\times\mathbf{p})}{\mathbf{k}^2}$$

$$+U_t\left(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \frac{3\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2}{\mathbf{k}^2}\right),$$

Coulomb potential does not run in QED



At LO,

$$\begin{aligned}
\nu \frac{dU_k}{d\nu} &= 0 \qquad\qquad \propto C_A \ \text{ in QCD} \\
\nu \frac{dU_2}{d\nu} &= \frac{2\alpha}{3\pi}\left(\frac{1}{m_1}+\frac{Z}{m_2}\right)^2 U_c + \frac{14 Z^2 \alpha^2}{3 m_1 m_2} = \gamma_0 U_c \\
\nu \frac{dU_3}{d\nu} &= \frac{2\alpha}{3\pi}\left(\frac{1}{m_1}+\frac{Z}{m_2}\right)^2 U_k + \gamma_1 U_c + \gamma_2 U_c^2
\end{aligned}$$

Exercise: Show that the ultrasoft gluon loop gives

$$\nu \frac{dV}{d\nu} = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1}+\frac{Z}{m_2}\right)^2 (\mathbf{p}-\mathbf{p}')^2 V(\mathbf{p},\mathbf{p}')$$

$$\gamma_0 = \frac{2\alpha}{3\pi}\left(\frac{1}{m_1^2} + \frac{Z}{4m_1m_2} + \frac{Z^2}{m_2^2}\right)$$

γ_0 is a constant in QED, since α does not run

Integrate:

$$U_2(\nu) = U_2(1) + \gamma_0 U_c \ln\nu$$

Only a single term, so the LO series terminates at $\alpha^5 \ln\alpha$.

No terms in $\alpha^4 (\alpha \ln\alpha)^n$ series except for $n = 1$. Power counting guarantees nothing left out at arbitrarily high orders.

$$
\begin{aligned}
\Delta E &= \langle U_2 \rangle \\
&= \gamma_0 U_c \ln \nu \, |\psi(0)|^2 \\
&= -\frac{8Z^4\alpha^5 m_R^3}{3\pi n^3}\left(\frac{1}{m_1^2} + \frac{Z}{4m_1 m_2} + \frac{Z^2}{m_2^2}\right) \ln Z\alpha,
\end{aligned}
$$

(Bethe 1947 for H)

$$|\psi(0)|^2 = \frac{(m_R Z\alpha)^3}{\pi n^3} \qquad n\text{S state}$$

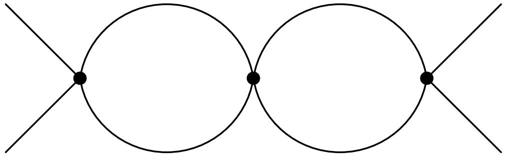
NLO

The interesting results are at NLO:

$$
\begin{aligned}
\nu \frac{dU_{2+s}}{d\nu}\bigg|_{\rm NLO} &= \rho_{ccc}\, U_c^3 + \rho_{cc2}\, U_c^2 \left(U_{2+s} + U_r\right) \\
&+\rho_{c22}\, U_c \left(U_{2+s}^2 + 2U_{2+s}U_r + \frac{3}{4}U_r^2 - 5U_t^2\mathbf{S^2}\right) \\
&+\rho_{ck}\, U_c U_k + \rho_{k2}\, U_k \left(U_{2+s} + U_r/2\right) \\
&+\rho_{c3}\, U_c \left(U_3 + U_{3s}S^2 + \frac{1}{2}U_{rk}\right),
\end{aligned}
$$

where $U_{2+s} = U_2 + U_s\mathbf{S}^2$ and $\rho_{c22} = -m_R^2/4\pi^2$.

NLO



Exercise: Compute the divergence in

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\;\frac{1}{\mathbf{k}^2}\;\frac{1}{(\mathbf{k}-\mathbf{q})^2}\;\frac{1}{\mathbf{q}^2}$$

$$\rho_{ccc} = -\frac{m_R^4}{64\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)^2, \qquad \rho_{c22} = -\frac{m_R^2}{4\pi^2},$$

$$\rho_{cc2} = -\frac{m_R^3}{8\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right), \qquad \rho_{c3} = \frac{2m_R}{\pi^2},$$

$$\rho_{ck} = \frac{m_R^2}{2\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right), \qquad \rho_{k2} = \frac{2m_R}{\pi^2}.$$

Integrating the NLO anomalous dimension–substitute LO values on the RHS for U_i.

Only terms which run at LO are U_2 and U_3

$$\int \text{const} = \ln \nu, \qquad \int \ln \nu = \frac{1}{2} \ln^2 \nu, \qquad \int \ln^2 \nu = \frac{1}{3} \ln^3 \nu$$

NLO series terminates after 3 terms

$$\alpha^6 \ln \alpha, \qquad \alpha^7 \ln^2 \alpha, \qquad \alpha^8 \ln^3 \alpha$$

Get the results for the Lamb shift, hyperfine splitting and widths for H, Ps, Muonium.

$\ln^3 \alpha$

$$\frac{1}{3}\,\gamma_0^2\,\rho_{c22}\,U_c^3(1)\,\ln^3\nu,$$

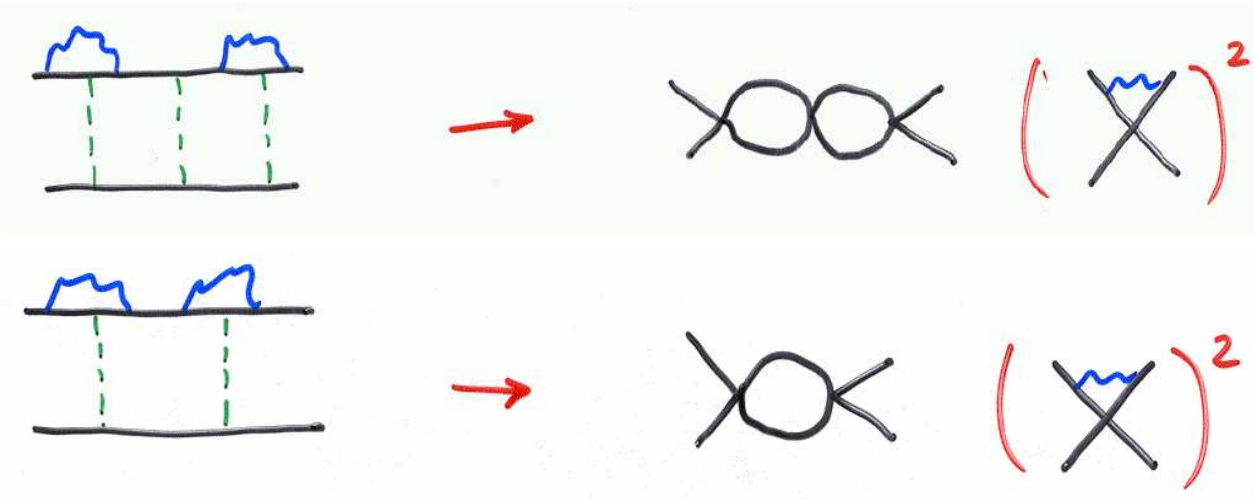
Lamb shift for the nS state (no HFS, Γ)

$$\begin{aligned}\Delta E &= \frac{64 m_R^5 \alpha^8 Z^6}{27\pi^2 n^3}\left(\frac{1}{m_1^2}+\frac{Z}{4m_1 m_2}+\frac{Z^2}{m_2^2}\right)^2 \ln^3(Z\alpha)\\ &= \frac{3 m_e \alpha^8 \ln^3\alpha}{8\pi^2 n^3} \quad \text{(positronium)}\end{aligned}$$

(8 KHz for Hydrogen 2P–2S)

Karshenboim 1993	$a_{63} = -8/27$
Malampalli and Sapirstein PRL 1998	$a_{63} = -0.652$
Goidenko et al. PRL 1999	$a_{63} = -0.296$
Yerokhin hep-ph/0001327	$a_{63} = -0.652$

$\ln^3 \alpha$



Odd dimensional integrals do not give anomalous dimensions:

$$\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{\mathbf{q}^2 + \mathbf{p}^2}{(\mathbf{q}-\mathbf{p})^2} \frac{1}{(\mathbf{q}-\mathbf{p}')^2}$$

$\ln^2 \alpha$

$$\gamma_0 \, \rho_{c22} \, U_c^2(1) \left[U_2(1) + U_s(1)\mathbf{S}^2 \right] \ln^2 \nu + \ldots$$

HFS: $$- \frac{64 Z^6 \alpha^7 m_R^5 \mu_1 \mu_2}{9 m_1 m_2 \pi n^3} \left[\frac{1}{m_1^2} + \frac{Z}{4 m_1 m_2} + \frac{Z^2}{m_2^2} \right] \ln^2(Z\alpha),$$

(Karshenboim 93, Labelle 94)

Ps HFS $$- \frac{7 m_e}{8 \pi n^3} \, \alpha^7 \ln^2 \alpha,$$

(Melnikov and Yelkhovsky 99) $\surd$

$$\frac{\Delta\Gamma}{\Gamma_0} = \gamma_0\,\rho_{c22}\,U_c(1)^2\ln^2\nu = -\frac{3}{2\pi}\alpha^3\ln^2\alpha,$$

(Karshenboim 93) $\surd$

Lamb Shift needs γ_1, γ_2

$\ln \alpha$

$$U_{2+s}\left[\rho_{c22}U_c\left(U_{2+s}+2U_r\right)+\rho_{cc2}U_c^2+\rho_{2k}U_k\right]\ln\nu+\ldots$$

$$\frac{\Delta\Gamma}{\Gamma_0}=\left(\frac{m_e^2}{2\pi}\mathsf{Re}\,U_{2+s}-2\right)\ln\nu=\left(\frac{7\mathbf{S}^2}{6}-2\right)\alpha^2\ln\alpha,$$

$$\left(\frac{\Delta\Gamma}{\Gamma_0}\right)_{\mathrm{ortho}}=\frac{\alpha^2}{3}\ln\alpha\,,\qquad\left(\frac{\Delta\Gamma}{\Gamma_0}\right)_{\mathrm{para}}=-2\alpha^2\ln\alpha\,,$$

(Caswell and Lepage 79, Khriplovich and Yelkhovsky 90) $\surd$

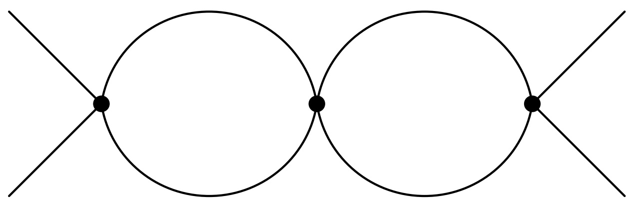
HFS and Lamb Shift depend on γ_3 and ρ_s

Velocity Renormalization Group

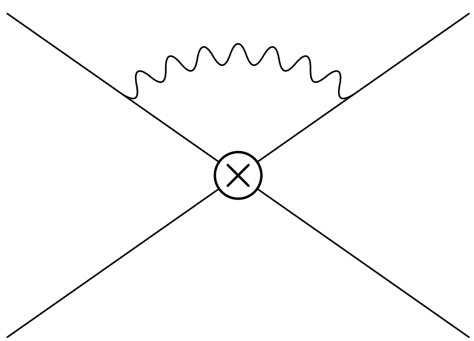
(Luke, Rothstein, A.M.; Stewart, A.M.)

<u>One-stage running:</u>

- Set $\mu_S = m\nu$, $\mu_U = m\nu^2$ and integrate from $\nu = 1$ to $\nu = v$.

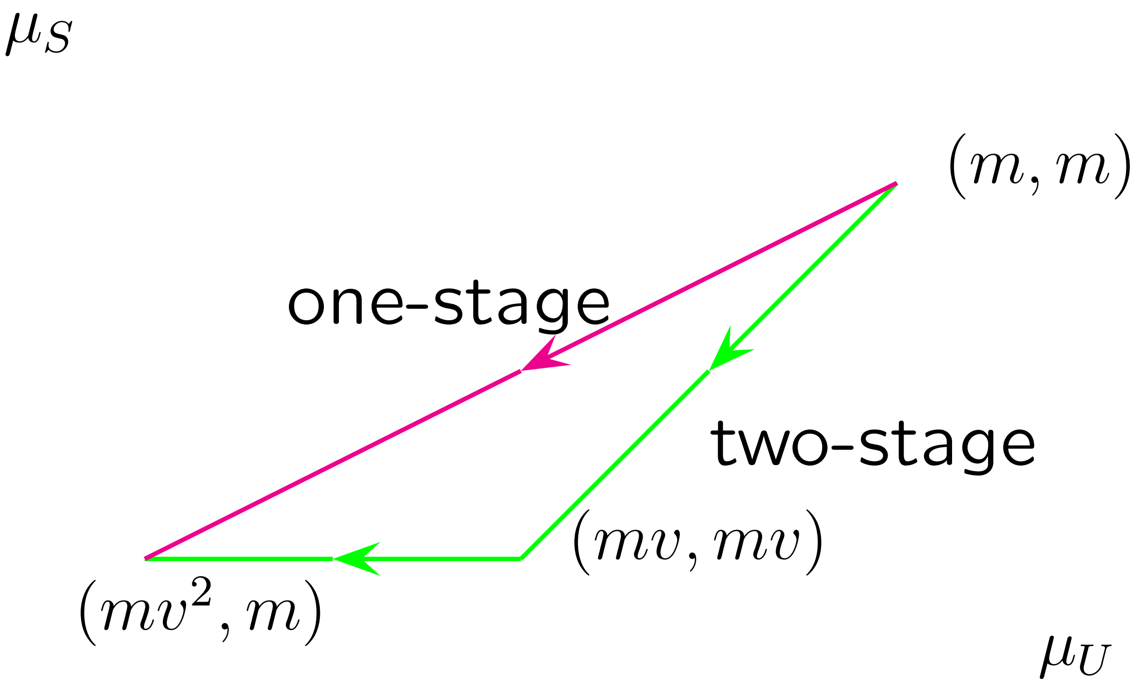


$$\ln \frac{\sqrt{mE}}{\mu_S} \sim \ln \frac{v}{\nu}$$



$$\ln \frac{E}{\mu_U} \sim \ln \frac{v^2}{\nu^2}$$

(Stewart, A.M.; Soto, Stewart, A.M.)

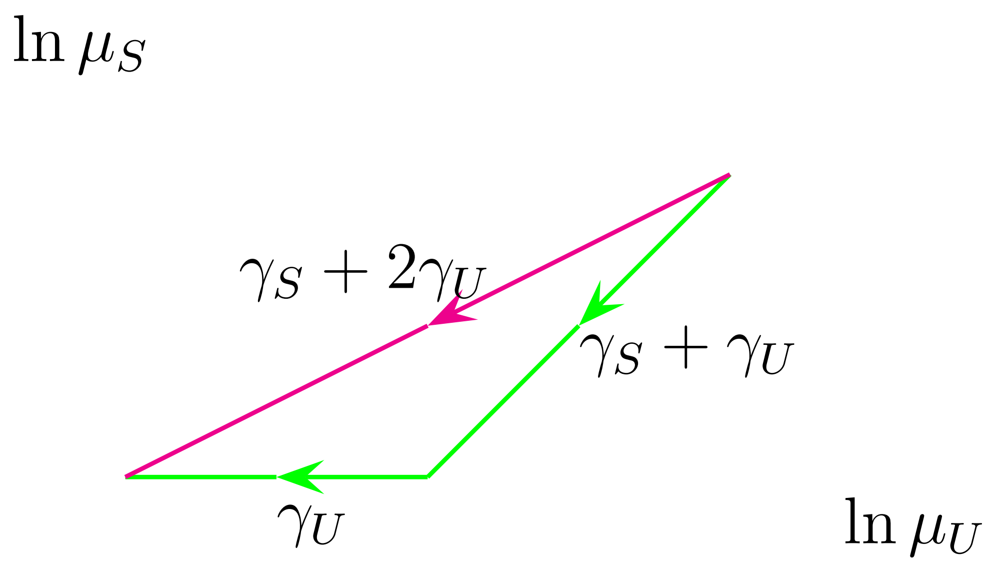


Two methods give different answers, $\nabla \times \gamma \neq \mathbf{0}$.

- One-stage VRG agrees with explicit QED calculations at $\alpha^7 \ln^2 \alpha$ and $\alpha^8 \ln^3 \alpha$
- Generic result for correlated scales: mv and mv^2 not independent.

RUN IN VELOCITY, NOT MOMENTUM

Define $\gamma_S = \frac{d}{d\ln\mu_S}$, $\gamma_U = \frac{d}{d\ln\mu_U}$



Two-stage		One-stage	
$\gamma_S + \gamma_U$	$m \to mv$	$\gamma_S + 2\gamma_U$	$1 \to v$
γ_U	$mv \to mv^2$		
$(\gamma_S + \gamma_U)\ln v + \gamma_U \ln v$		$(\gamma_S + 2\gamma_U)\ln v$	

Single-log terms agree. BUT γ not constants, and depend on couplings, $\gamma(V)$, and V can run.

So one gets:

$$
\begin{array}{ll}
\gamma_S\,(\gamma_U)\ln^2 v & \qquad \gamma_S(2\gamma_U)\ln^2 v \\
\gamma_S\,(\gamma_U)^2\ln^3 v & \qquad \gamma_S(2\gamma_U)^2\ln^3 v
\end{array}
$$

and the two methods differ.

Other results

Extended the analysis to QCD where α runs. See that computations involve both $\alpha_s(mv)$ and $\alpha_s(mv^2)$.

1. Running V.
2. QCD Lamb shift. The series no longer terminate since α_s runs.
3. $\bar{t}t$ production near threshold.

Potentials for $t\bar{t}$

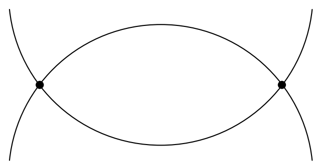
(Hoang, Stewart, A.M.)

	$\mathcal{V}_r^{(s)}$	$\mathcal{V}_2^{(s)}$	$\mathcal{V}_s^{(s)}$	$\mathcal{V}_\Lambda^{(s)}$	$\mathcal{V}_t^{(s)}$
$\nu = 1$	-1.81	0	0.60	0.15	2.71
$\nu = v$	-1.39	0.61	0.53	0.16	3.11

$$
\begin{aligned}
\mathcal{V}_2^{(1)}(\nu) \quad = \quad & \frac{4\pi C_1}{\beta_0}\,[\alpha_s(m) - \alpha_s(m\nu)]\ln\left(\frac{m\nu}{m}\right) \\
& - \frac{16\pi C_1}{3\beta_0}\,\alpha_s(m)\ln\left[\frac{\alpha_s(m\nu)}{\alpha_s(m\nu^2)}\right]
\end{aligned}
$$

Summary

1. Systematic v expansion for bound states.
2. Can sum logarithms using the VRG.
3. Technical issues having to do with implementing the soft sector not understood.
4. Can test various field theory ideas on a "non-perturbative" system that can be solved exactly. e.g. off-shell potentials and the use of the equations of motion. $V_\Delta \to V_k$ converted using perturbative graph with $\mathbf{p}^2 = \mathbf{p}'^2$



but also true for the matrix element in the Coulomb wave function.

Experimental Situation

		Expt.(MHz)	Theory(MHz)	Agree?
H	Lamb	1057.845(9)	1057.833(6)	$< r_p^2 >$
			1057.814(6)	
	h.f.s	1420.4057517667(9)	1420.399(2)	G_E, G_M
$\mu^+ e^-$	h.f.s	4463.302765(53)	4463.30267(27)	m_e/m_μ
$e^+ e^-$	Lamb	13012.4(1)	13012.41(8)	agree
	h.f.s	203389.10(74)	203391.69(16)	3σ
	$\Gamma_{\rm para}$	7990.9(1.7) μs^{-1}	7989.620(13) μs^{-1}	agree
	$\Gamma_{\rm ortho}$	7.0398(29) μs^{-1}	7.039968(10) μs^{-1}	6-9σ(?)
		7.0482(16) μs^{-1}		
		7.0514(14) μs^{-1}		