Historical and other remarks on effective theories

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Effective Theories in Particle Physics

Zuoz, July 18, 2006

History of effective theory of strong interaction

biased account, my perspective, not a historian
only discuss low energy effective theory of the strong interaction
wisdom possibly grows with age, but memory does not . . .

- effective theory of the strong interaction was born before the theory of the strong interaction
- crucial element: hidden approximate symmetry
  pions play special role: Goldstone bosons
  would be massless if the hidden symmetry were exact

Nambu 1960

life at our institute at that time

- *Milan Locher & I were studying physics, but did not hear much about particles – beyond QED, Fermi theory of β decay and nonrelativistic potential models for the nuclear forces*

  There was nothing to miss. The theory of the strong interaction was a genuine mess: elementary fields for baryons and mesons, Yukawa interaction for the strong forces, perturbation theory with coupling constants of order 1, nuclear democracy, bootstrap . . . absolutely nothing worked even half ways, beyond general principles like Lorentz invariance, causality, unitarity, crossing, dispersion relations

  smart people considered Regge theory very promising

  Veneziano model 1968

- *Mme Tonnelat (Inst. Poincaré, Paris) was guest professor at the University of Bern, lectured on gravity, unification of gravity & electromagnetism, Einstein - Schrödinger, Kaluza - Klein, . . .*

  she was enthusiastic about the progress in observational cosmology, expected the deceleration parameter q in Hubble's law to be measured within one or at most two years . . .

what I did at that time

- we were asked to give our seminars en français, ou, faute de mieux, en français fédéral *fille*
- wrote my diploma thesis on Kaluza-Klein (1960)
- in France, a decent way to communicate scientific results was to submit these to a member of the Académie des Sciences

RELATIVITÉ. — *Sur une modification des théories pentadimensionnelles destinée à éviter certaines difficultés de la théorie de Jordan-Thiry.* Note (*) de M. **Heinrich Leutwyler**, présentée par M. Louis de Broglie.

Les équations du champ adoptées par Y. Thiry [1]

$$S_{\alpha\beta} = r u_\alpha u_\beta \qquad (\alpha, \beta = 0, 1, 2, 3, 4;\ i, k = 1, 2, 3, 4;\ x^4 = ct) \tag{1}$$

conduisent à des difficultés dans la définition d'une variation à symétrie sphérique [2] et aussi dans l'obtention d'équations approchées du mouvement [3].

En effet K. Just a montré que l'application de (1) au calcul du champ de gravitation créé par une masse neutre possédant la symétrie sphérique conduisait à modifier d'un facteur 5/4 la valeur prévue pour l'avance du périhélie de Mercure. Cette conclusion est indépendante du choix de

Comptes rendus des séances de l'Académie des Sciences, séance du 21 novembre 1960

current algebra

- eightfold way — Gell-Mann, Ne'eman 1961
- pattern of symmetry breaking, Ω^- — Gell-Mann, Okubo 1961/1962
- quark model — Gell-Mann, Zweig 1962
- puzzle: why is the symmetry not exact ?
  exact consequences of approximate properties ?
  charges & currents form an exact algebra
  even if they do not commute with the hamiltonian

  Gell-Mann 1964
- test of current algebra: size of $\langle N|A^\mu|N\rangle \sim g_A$

  Adler 1965, Weisberger 1966
- prediction from current algebra: $\pi\pi$ scattering lengths

  Weinberg 1966

mass of the pion

- formula for pion mass

$$M_\pi^2 = \underbrace{(m_u + m_d)}_{\Uparrow \text{ explicit}} \times \,|\langle 0|\, \underbrace{\bar{q}\, q}_{\Uparrow \text{ spontaneous}} \,|0\rangle| \times \frac{1}{F_\pi^2}$$

Gell-Mann, Oakes & Renner 1968

  - *at that time, the existence of quarks was questionable*
  - *quarks were treated like the bread used to prepare a pheasant in the royal French cuisine*

$\Rightarrow$ *formula does not appear like this in the paper*

quark masses

- even before the discovery of QCD, attempts at estimating the masses of the quarks were made
- bound state models for mesons and baryons
  $m_u + m_u + m_d \simeq M_p \qquad m_u \simeq m_d$

$\Rightarrow m_u \simeq m_d \simeq 300$ MeV "constituent masses"

- remarkably simple and successful picture explains the pattern of energy levels without QCD
- model for spontaneous symmetry breakdown requires much smaller fermion masses

  Nambu & Jona-Lasinio 1961
- same conclusion from sum rules for currents

  Okubo 1969
- conceptual basis of royal French cuisine ?

QCD

- QCD was discovered in 1973
  - many considered this a wild speculation
  - all quantum field theories encountered in nature so far had the spectrum of perturbation theory Pauli
  - also true of the electroweak theory
    Glashow 1961, Weinberg 1967, Salam 1968
- only gradually, particle physicists abandoned their outposts in no man's and no woman's land, returned to the quantum fields and resumed discussion in the good old Gasthaus zu Lagrange Jost

$\Rightarrow$ Standard Model, clarified the picture enormously

Standard Model

Standard Model appeared like a miracle:

- weak, e.m. and strong interactions are very different
  nevertheless, they are all generated by gauge fields

  IG Physik, Gesellschaft mit besonderer Haftung, advertisement ca. 1973

  Im Falle eines Falles
  klebt ein EICHFELD
  wirklich alles !

  Bezugsquellennachweis

  H. Weyl, Z. Phys. 56 (1929) 330, C. N .Yang and R. Mills, Phys. Rev. 96 (1954) 191

- gauge fields are renormalizable in $d = 4$
- paradigm has changed: SM cannot be the full truth
  no reason for an effective theory to be renormalizable

$\Rightarrow$ why is the SM renormalizable ?

pattern of light quark masses

- SU(6) model for the wave functions of π, K, ρ

$$\frac{(m_u + m_d)}{2} = \frac{F_\pi M_\pi^2}{3 F_\rho M_\rho} \simeq 5\,\text{MeV}, \qquad m_s \simeq 135\,\text{MeV}$$

"Is the quark mass as small as 5 MeV ?" L. 1974

- difference between m_u and m_d ?
  reanalyzed the Cottingham formula Gasser & L. 1975

$\Rightarrow$ e.m. self energy of proton $>$ neutron

$\Rightarrow$ $M_p < M_n$ cannot be due to the e.m. interaction

$\Rightarrow$ $M_p < M_n$ must be due to $m_u < m_d$

$\Rightarrow$ isospin not a symmetry of the strong interaction !

pattern of light quark masses

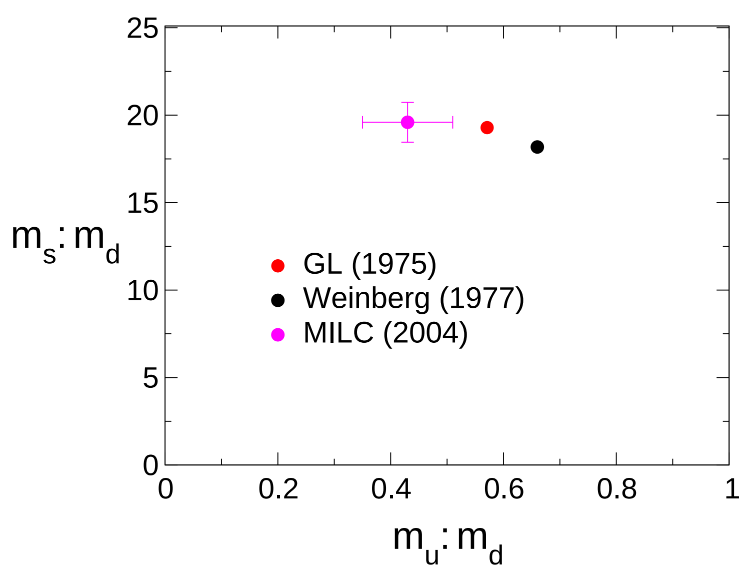
- $m_u \simeq 4\,\text{MeV}, \;\; m_d \simeq 7\,\text{MeV}, \;\; m_s \simeq 135\,\text{MeV}$

  Gasser & L. 1975; Zuoz lecture notes 1975

  - m_u and m_d are very different
  - m_u and m_d are small compared to m_s
  - "constituent masses" $\notin$ lagrangian of QCD
    vague, model dependent notion

- took quite a while before this pattern was taken seriously extra muros

  Weinberg 1977



approximate flavour symmetries of QCD

- why is isospin such a good quantum number ?
  QCD has an intrinsic scale ~ 1 GeV

  dimensional transmutation, divergences of perturbation theory $\in$ physics

  $m_d - m_u \ll$ scale of QCD, not $\ll m_u + m_d$
- why is eightfold way a decent approximate symmetry ?
  $m_s - m_u \ll$ scale of QCD
- isospin is an even better symmetry because
  $m_d - m_u \ll m_s - m_u$
- approximate symmetries are natural for QCD
- $m_u \ll m_s \Rightarrow m_u, m_d, m_s \ll$ scale of QCD

$\Rightarrow$ masses of the light quarks represent perturbations
can neglect these in a first approximation

massless quarks

- as far as the strong interaction goes, the only difference between u, d, s is the mass
- for massless fermions, the right- and left-handed components lead a life of their own $\Rightarrow$ chiral symmetry
- fictitious world with $m_u = m_d = m_s = 0$:
  QCD acquires an exact chiral symmetry
  no distinction between u_L, d_L, s_L, nor between u_R, d_R, s_R
  hamiltonian is invariant under SU(3)$_L \times$ SU(3)$_R$
- chiral symmetry is hidden, "spontaneously broken":
  ground state is not symmetric under SU(3)$_L \times$ SU(3)$_R$
  symmetric only under the subgroup SU(3) = SU(3)$_{L+R}$

$\Rightarrow$ mesons and baryons form degenerate SU(3) multiplets
the lowest multiplet is massless:
$M_{\pi^\pm} = M_{\pi^0} = M_{K^\pm} = M_{K^0} = M_{\bar{K}^0} = M_\eta = 0$
Goldstone bosons of the hidden symmetry

strength of strong interaction at low energies

- strength of the interaction fixed by F_π

$$g_{\pi NN} = \frac{g_A\, M_N}{F_\pi} \quad \pi N \text{ interaction}$$ Goldberger & Treiman 1958

in massless QCD, this relation is exact

$$A(s,t,u) = \frac{s}{F_\pi^2} + O(p^4) \quad \pi\pi \text{ interaction}$$ Weinberg 1966

leading term in expansion in powers of momenta

$\Rightarrow$ pions of zero momentum do not interact
only interact weakly if momenta are small

$\Rightarrow$ at low temperature and for $m_u = m_d = m_s = 0$,
hadronic matter is a free gas of Goldstone bosons:

energy density = $\dfrac{4\pi^2}{15} T^4 + O(T^8)$ = $3 \times$ pressure

effective theory

- it is crucial that Goldstone bosons of low momentum interact only weakly: can treat the momenta as well as the quark masses as perturbations

$\Rightarrow$ chiral perturbation theory

- formulation in terms of an effective lagrangian

Weinberg 1967, Coleman, Wess, Zumino, Callan, Dashen, Weinstein 1969

- unperturbed lagrangian describes massless GB

$\Rightarrow$ chiral perturbation series has infrared singularities

Li & Pagels 1971, Langacker & Pagels 1973, Gasser & Zepeda 1980

- example: expand M_π^2 in powers of the quark masses

$M^2 \equiv (m_u + m_d)B$ GMOR

$$M_\pi^2 = M^2 + \frac{M^4}{32\pi^2 F_\pi^2} \ln \frac{M^2}{\Lambda_3^2} + O(M^6)$$

expansion is not a Taylor series

chiral perturbation theory

- some of our estimates for the quark masses relied on leading order mass formulae for mesons or baryons
- higher orders in the expansion $\Rightarrow$ nonanalytic terms ?
  χPT provides controlled framework, also for σ term

  This was my motivation for studying χPT   Gasser 1981, Gasser & L. 1982
- χPT originally formulated as a meson field theory
  - $\langle 0|\, T\pi^i(x)\pi^k(y)\,|0\rangle$ plays central role
  - depends on choice of variables, but the result for meson masses, S-matrix is unambiguous
  - studying the Green functions of the pion field amounts to perturbing the system with

    $\mathcal{L}_{eff} \longrightarrow \mathcal{L}_{eff} + \vec{f}(x)\cdot\vec{\pi}(x)$

    $\vec{\pi}(x)$ transforms in nonlinear manner

$\Rightarrow$ ruins the symmetry of the effective lagrangian

effective theory for QCD Green functions

- further shortcoming of original framework: current matrix elements ? Noether currents of $\mathcal{L}_{eff}$ are correct only at leading order, F_π at NLO ?
- $\vec{\pi}(x) \notin$ QCD, but $\vec{V}^\mu(x),\ \vec{A}^\mu(x), \ldots \in$ QCD
  $\mathcal{L}_{\text{QCD}} \longrightarrow \mathcal{L}_{\text{QCD}} + \vec{f}(x) \cdot \vec{\pi}(x)$?
  $\mathcal{L}_{\text{QCD}} \longrightarrow \mathcal{L}_{\text{QCD}} + \vec{f}_\mu(x) \cdot \vec{V}^\mu(x) + \ldots$ $\surd$
- need effective theory for Green functions of QCD
  $\mathcal{L}_{eff} \longrightarrow \mathcal{L}_{eff} + \vec{f}_\mu(x) \cdot \vec{V}^\mu_{eff}(x) + \ldots$
- can express the symmetry through the Green functions:
  symmetry $\Rightarrow$ current conservation $\Rightarrow$ Ward identities
  WI remain exact even for $m_u, m_d, m_s \neq 0$
  anomalies show up in WI, not in lagrangian

Gasser & L. 1984, 1985

plethora of effective coupling constants

- in principle, the effective theory is exact
  yields expansion of QCD Green functions in p, m_q
- χPT merely exploits the symmetries of QCD:
  yields the general solution of the Ward identities
  symmetries only relate – do not determine
- number of terms in $\mathcal{L}_{eff}$ rapidly grows with order:
  LO: 2, NLO: 10, NNLO: 90
- further effective coupling constants needed for low energy analysis of the e.m. and weak interactions

masses of the Goldstone bosons at LO

- LO: meson masses fix quark mass ratios

$$\frac{m_u}{m_d}=\frac{M^2_{\pi^+}-M^2_{K^0}+M^2_{K^+}}{M^2_{\pi^+}+M^2_{K^0}-M^2_{K^+}}\left\{1+O(m_q)\right\}$$

$$\frac{m_s}{m_d}=\frac{M^2_{K^0}+M^2_{K^+}-M^2_{\pi^+}}{M^2_{K^0}-M^2_{K^+}+M^2_{\pi^+}}\left\{1+O(m_q)\right\}$$

numerically: $\frac{m_u}{m_d}\simeq 0.66 \qquad \frac{m_s}{m_d}\simeq 20$

Weinberg 1977

- LO: symmetry imposes constraint on meson masses

$$M^2_\eta=\frac{4}{3}M^2_K-\frac{1}{3}M^2_\pi$$

Gell-Mann-Okubo formula

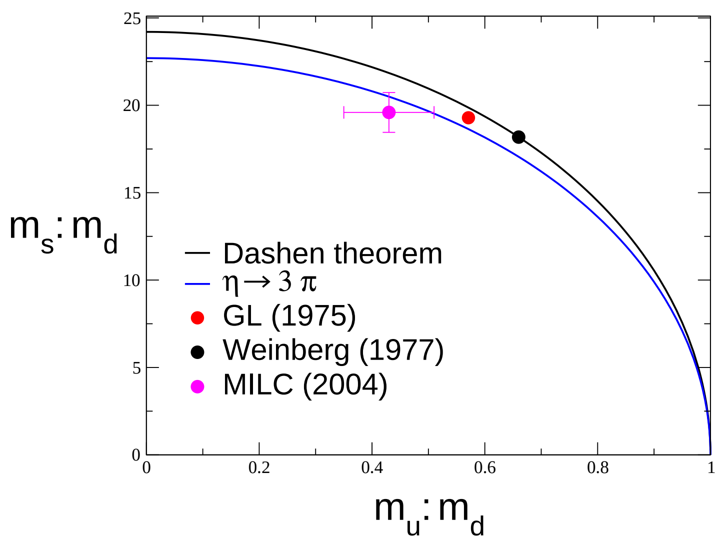
phenomenological ambiguity at NLO

- NLO: result for meson masses involves 3 new coupling constants (L_6, L_7, L_8)

$\Rightarrow$ GMO formula not valid, no constraint on meson masses

$\Rightarrow$ cannot extract $m_u : m_d$ and $m_s : m_d$ separately
NLO mass formulae only correlate the two ratios

Kaplan & Manohar 1986



- position on ellipse depends on L_8
- formally, can even have $m_u = 0$

$\Rightarrow$ QCD would be CP-invariant
way out of the strong CP-puzzle ?

- requires enormous NLO corrections at LO, K^0-K^+ is 4 times too large

"I conclude that m_u = 0 is an interesting way not to understand this world – it is not the only one." L. 1990

expansion in powers of m_q

- cannot vary the quark masses experimentally

$\Rightarrow$ not all of the coupling constants can be measured
need theoretical estimates for the remaining ones
large N_c, sum rules, lattice results (L_4, L_6), ...

- truncated perturbation series is meaningful only if the effective coupling constants are not too large

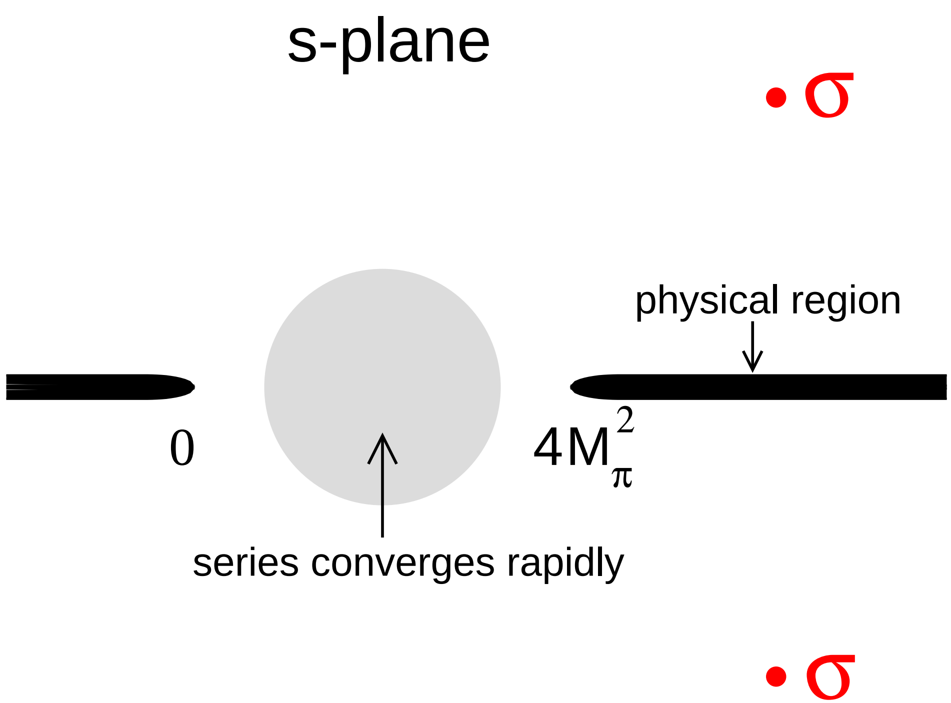
$\Rightarrow$ cannot treat the coupling constants as free parameters
LO + corrections

- couplings at LO and NLO are now known quite well
LO contributions indeed dominate meson masses
no surprises: $\langle 0|\,\bar{q}q\,|0\rangle$ not small, L_6, L_7, L_8 not large

$\Rightarrow$ in the range $0 \leq m_u, m_d, m_s \leq$ physical values:
$\mathcal{L}_{eff}$ is approximately linear in m_u, m_d, m_s
compare $\mathcal{L}_{\text{QCD}}$: linear in all quark masses

expansion in powers of momenta

- poles and cuts from Goldstone boson exchange dominate the QCD Green functions at low energies
- resonances (σ, ρ, $\ldots$) $\notin$ domain of validity of χPT
- example: $\pi\pi$ scattering amplitude



domain can be extended with dispersion theory

$\pi\pi$ interaction now known very well at low energies

$\Rightarrow$ rest of the talk

$\pi\pi$ interaction

- plays a crucial role whenever the strong interaction is involved at low energies

  example: Standard model prediction for muon magnetic moment

- main experiments on $\pi\pi$ scattering were done in the seventies – what's new ?

  - significant theoretical progress, based on χPT + dispersion theory
  - new precision data:

$K \to \pi\pi\ell\nu$	E865	Brookhaven
pionic atoms	DIRAC	CERN
$K \to 3\pi$	NA48/2	CERN

  - lattice results on M_π, F_π, a_0^2, $\langle r^2 \rangle_s$

analyticity and crossing

- $\pi\pi$ scattering is special: crossed channels are identical

$\Rightarrow$ Re $T(s,t)$ can be represented as a twice subtracted dispersion integral over Im $T(s,t)$ in physical region

S.M. Roy 1971

- the 2 subtraction constants can be identified with the S-wave scattering lengths:

$$a_0^0\ ,\ a_0^2 \begin{array}{l}\leftarrow \text{isospin}\\ \leftarrow \text{angular momentum}\end{array}$$

- representation leads to dispersion relations for the individual partial waves: *Roy equations*

Roy equations

- pioneering work on the physics of the Roy equations: Basdevant, Froggatt & Petersen 1974
- dispersion integrals converge rapidly (2 subtractions)

$\Rightarrow$ crude phenomenological information on Im $T(s,t)$ for energies above 800 MeV suffices

$\Rightarrow$ given a_0^0, a_0^2, the scattering amplitude can be calculated to within small uncertainties

Ananthanarayan, Colangelo, Gasser & L. 2001
Descotes, Fuchs, Girlanda & Stern 2002

$\Rightarrow$ a_0^0, a_0^2 are the essential parameters at low energy

- main problem in early work: a_0^0, a_0^2 poorly known experimental information near threshold is meagre

low energy theorems

- chiral perturbation theory provides the missing piece: theoretical prediction for a_0^0, a_0^2

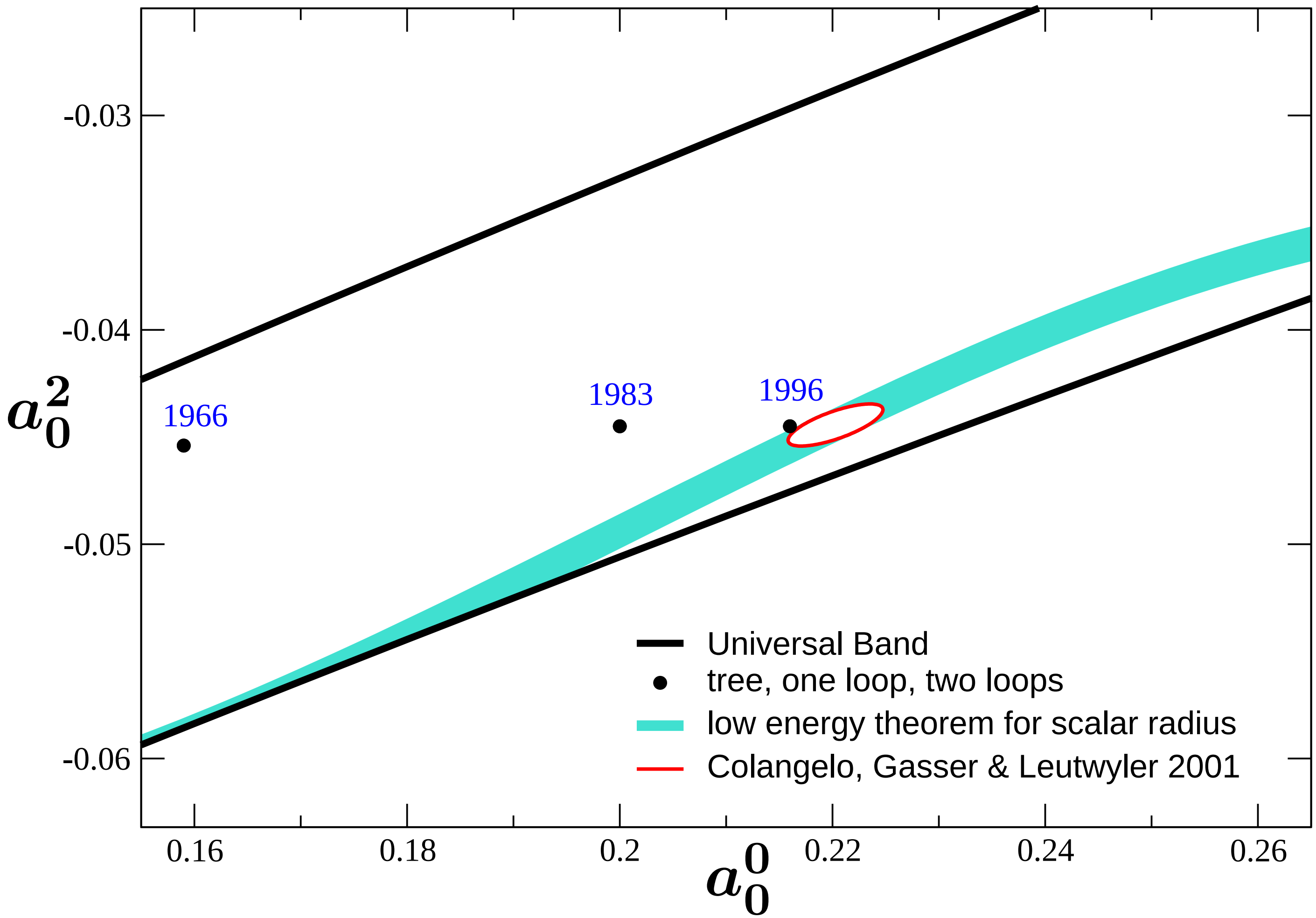
  Weinberg 1966, Gasser & L. 1983, Bijnens, Colangelo, Ecker, Gasser & Sainio 1996

- most accurate results for a_0^0, a_0^2 are obtained by matching the chiral and dispersive representations near the center of the Mandelstam triangle

  Colangelo, Gasser & L. 2001

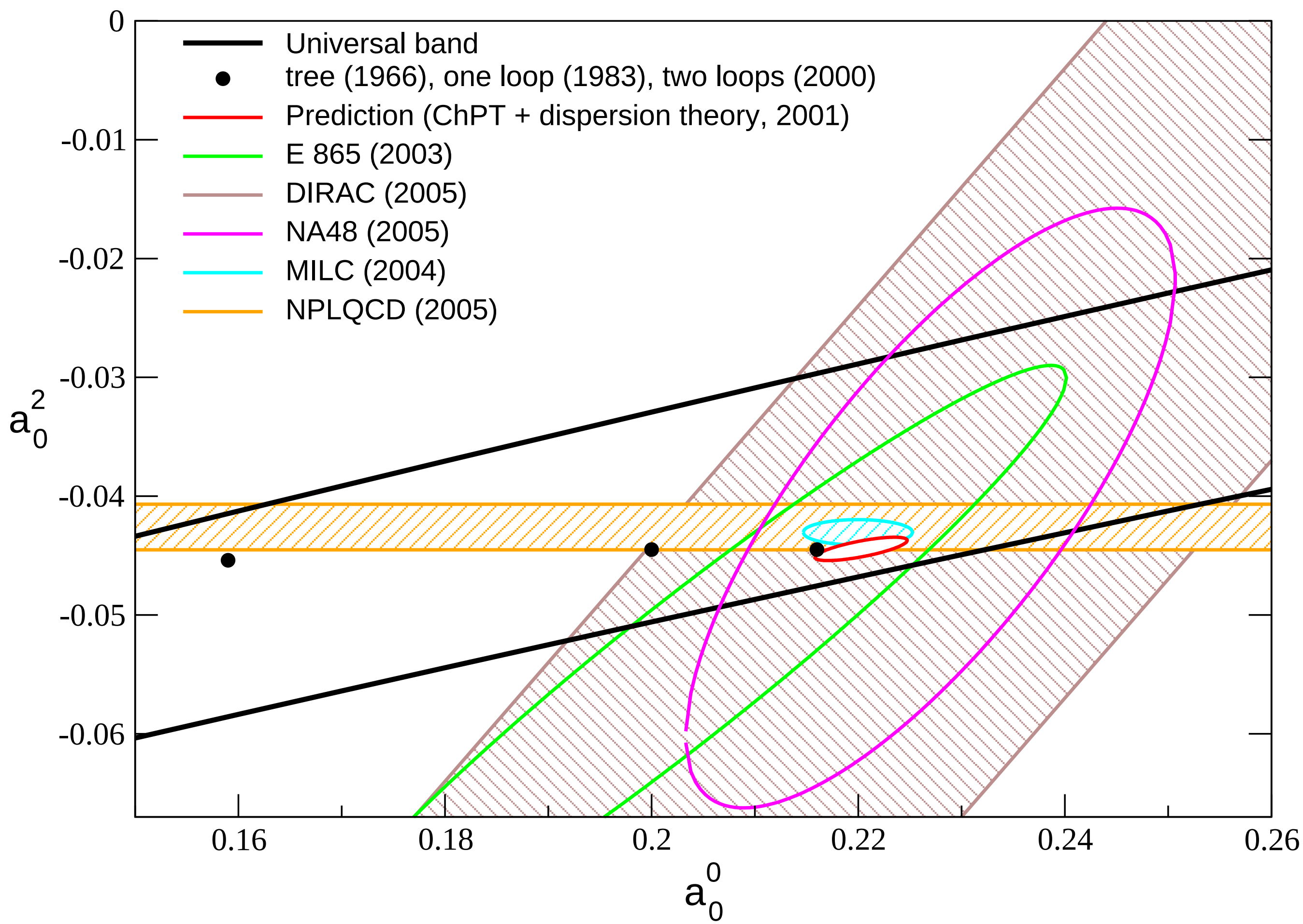
- in combination with the low energy theorems for a_0^0, a_0^2, the dispersion relations for the partial waves fix the $\pi\pi$ scattering amplitude to an incredible degree of accuracy

predictions for the S-wave $\pi\pi$ scattering lengths

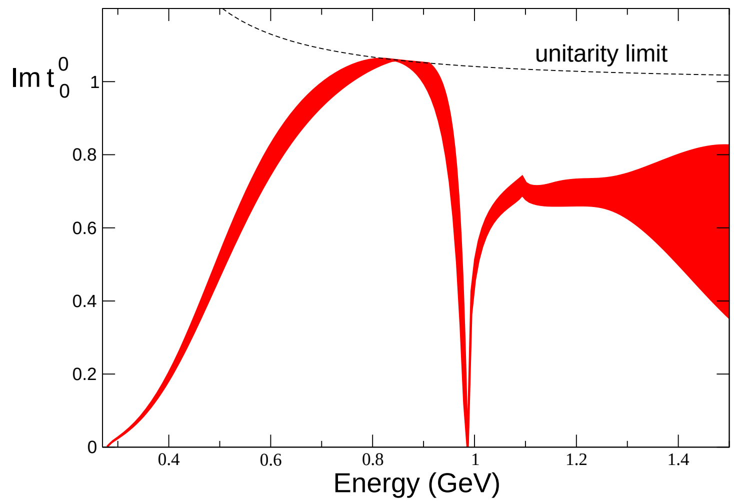


sizeable corrections in a_0^0, while a_0^2 nearly stays put

tests of the predictions for a_0^0, a_0^2: experiment & lattice



theory is ahead of experiment . . .



There is the broad object seen in $\pi\pi$ *scattering, often called "background", which extends from about 400 MeV up to about 1700 MeV. This object we consider as a single broad resonance*[2] *which we identify as the lightest glueball with quantum numbers* $J^{PC} = 0^{++}$ *...*

[2] *we refer to it as red dragon*

P. Minkowski & W. Ochs, Eur. Phys. J. C9 (1999) 283

the red dragon

I. Caprini, G. Colangelo & H. Leutwyler, Phys. Rev. Lett. 96 (2006) 132001

- does QCD have a resonance near threshold ?
- why care ?
  - concerns the non-perturbative domain of QCD
  - quark and gluon degrees of freedom useless there

  $\Rightarrow$ understanding very poor, pattern of energy levels ?
  - lowest resonance: σ ? ρ ?
- resonance $\leftrightarrow$ pole on second sheet
  - poles are universal
  - pole position is unambiguous, even if width is large
  - where is the pole closest to the origin ?

Citation: S. Eidelman *et al.* (Particle Data Group), Phys. Lett. B **592**, 1 (2004) and 2005 partial update for edition 2006 (URL: http://pdg.lbl.gov)

$f_0(600)$ or σ

$I^G(J^{PC}) = 0^+(0^{++})$

A REVIEW GOES HERE – Check our WWW List of Reviews

$f_0(600)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\ \mathrm{Im}(\sqrt{s_{\mathrm{pole}}})$.

VALUE (MeV)	*DOCUMENT ID*		*TECN*	*COMMENT*
(400–1200)–i(300–500) OUR ESTIMATE				
• • • We do not use the following data for averages, fits, limits, etc. • • •				
$(541 \pm 39)-i(252 \pm 42)$	[1] ABLIKIM	04A	BES2	$J/\psi \to \omega\pi^+\pi^-$
$(528 \pm 32)-i(207 \pm 23)$	[2] GALLEGOS	04	RVUE	Compilation
$(440 \pm 8)-i(212 \pm 15)$	[3] PELAEZ	04A	RVUE	$\pi\pi \to \pi\pi$
$(533 \pm 25)-i(247 \pm 25)$	[4] BUGG	03	RVUE	
$532 - i272$	BLACK	01	RVUE	$\pi^0\pi^0 \to \pi^0\pi^0$
$(470 \pm 30)-i(295 \pm 20)$	[5] COLANGELO	01	RVUE	$\pi\pi \to \pi\pi$
$(535^{+48}_{-36})-i(155^{+76}_{-53})$	[6] ISHIDA	01		$\Upsilon(3S) \to \Upsilon\pi\pi$
$610 \pm 14 - i620 \pm 26$	[7] SUROVTSEV	01	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(558^{+34}_{-27})-i(196^{+32}_{-41})$	ISHIDA	00B		$p\overline{p} \to \pi^0\pi^0\pi^0$
$445 - i235$	HANNAH	99	RVUE	π scalar form factor
$(523 \pm 12)-i(259 \pm 7)$	KAMINSKI	99	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\sigma\sigma$
$442 - i\ 227$	OLLER	99	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$469 - i203$	OLLER	99B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$445 - i221$	OLLER	99C	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\eta\eta$
$(1530^{+\ 90}_{-250})-i(560 \pm 40)$	ANISOVICH	98B	RVUE	Compilation
$420 - i\ 212$	LOCHER	98	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(602 \pm 26)-i(196 \pm 27)$	[8] ISHIDA	97		$\pi\pi \to \pi\pi$
$(537 \pm 20)-i(250 \pm 17)$	[9] KAMINSKI	97B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, 4π
$470 - i250$	[10,11] TORNQVIST	96	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $K\pi$, $\eta\pi$
$\sim (1100 - i300)$	AMSLER	95B	CBAR	$\overline{p}p \to 3\pi^0$
$400 - i500$	[11,12] AMSLER	95D	CBAR	$\overline{p}p \to 3\pi^0$
$1100 - i137$	[11,13] AMSLER	95D	CBAR	$\overline{p}p \to 3\pi^0$
$387 - i305$	[11,14] JANSSEN	95	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$525 - i269$	[15] ACHASOV	94	RVUE	$\pi\pi \to \pi\pi$
$(506 \pm 10)-i(247 \pm 3)$	KAMINSKI	94	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$370 - i356$	[16] ZOU	94B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$408 - i342$	[11,16] ZOU	93	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$870 - i370$	[11,17] AU	87	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$470 - i208$	[18] BEVEREN	86	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\eta\eta$, ...
$(750 \pm 50)-i(450 \pm 50)$	[19] ESTABROOKS	79	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(660 \pm 100)-i(320 \pm 70)$	PROTOPOP...	73	HBC	$\pi\pi \to \pi\pi$, $K\overline{K}$
$650 - i370$	[20] BASDEVANT	72	RVUE	$\pi\pi \to \pi\pi$

model independent determination of the pole

- all of the results quoted by the PDG are obtained by
  (a) parametrizing the data for real values of s
  (b) continuing this parametrization analytically in s

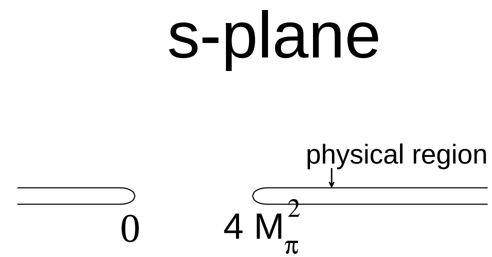
  $\Rightarrow$ result is sensitive to the parametrization used

- we found a model independent method:
  1. poles on second sheet are zeros on first sheet
  2. the Roy equations are valid for complex values of s, in a limited region of the first sheet

  $\Rightarrow$ exact representation of the partial waves in terms of observable quantities, valid for complex values of s

  3. can evaluate this representation to good precision and determine the zeros numerically

pole on second sheet $\leftrightarrow$ zero on first sheet



- $S_0^0(s) = \eta_0^0(s) \exp 2i\delta_0^0(s)$

  $S_0^0(s)$ is analytic in the cut plane

- for $0 < s < 4M_\pi^2$, $S_0^0(s)$ is real

$\Rightarrow S_0^0(s^\star) = S_0^0(s)^\star$

x in elastic interval: $S_0^0(x \pm i\epsilon) = \exp \pm 2i\delta_0^0(x)$

- second sheet is reached by continuation across the elastic interval of the right hand cut

  $S_0^0(x - i\epsilon)^{II} = S_0^0(x + i\epsilon)^{I} = 1/S_0^0(x - i\epsilon)^{I}$

  analyticity $\Rightarrow$ $\boxed{S_0^0(s)^{II} = 1/S_0^0(s)^{I}}$ valid $\forall s$

  pole in $S_0^0(s)^{II} \Longleftrightarrow$ zero in $S_0^0(s)^{I}$

Roy equation for the isoscalar S-wave

$$S_0^0(s) = 1 + 2\, i\rho\, t_0^0(s) \qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\infty} ds' \, \{K_0(s,s')\, \mathrm{Im}\, t_0^0(s')$$

$$+ K_1(s,s')\, \mathrm{Im}\, t_1^1(s') + K_2(s,s')\, \mathrm{Im}\, t_0^2(s')\}$$

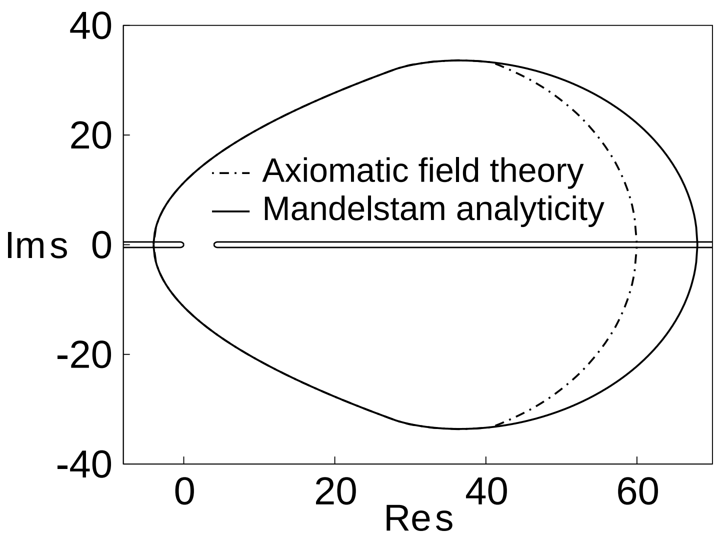
$+$ higher partial waves

- the subtraction constants are determined by a_0^0, a_0^2:
  $a = a_0^0\,, \qquad b = (2a_0^0 - 5a_0^2)/(12M_\pi^2)$
- the kernels are elementary functions, e.g.
  $$K_0(s,s') = \underbrace{\frac{1}{\pi(s'-s)}}_{r.h.cut} + \underbrace{\frac{2\, ln\{(s+s'-4M_\pi^2)/s'\}}{3\pi(s-4M_\pi^2)} - \frac{5s'+2s-16M_\pi^2}{3\pi s'(s'-4M_\pi^2)}}_{l.h.cut}$$
- left hand cut is essential for convergence:
  $K_0(s,s') \sim 1/{s'}^3$ for large s'

domain of validity of the Roy equations

- Roy derived his equations for real energies in the interval $-4M_\pi^2 < s < 60M_\pi^2$
- equations are valid for complex s in a limited region of the first sheet

I. Caprini, G. Colangelo & H. Leutwyler, Phys. Rev. Lett. 96 (2006) 132001



s in units of M_π^2

- proof is based on first principles, general quantum field theory

A. Martin, *Scattering Theory: Unitarity, Analyticity and Crossing*, Lecture Notes in Physics, vol. 3, 1969.

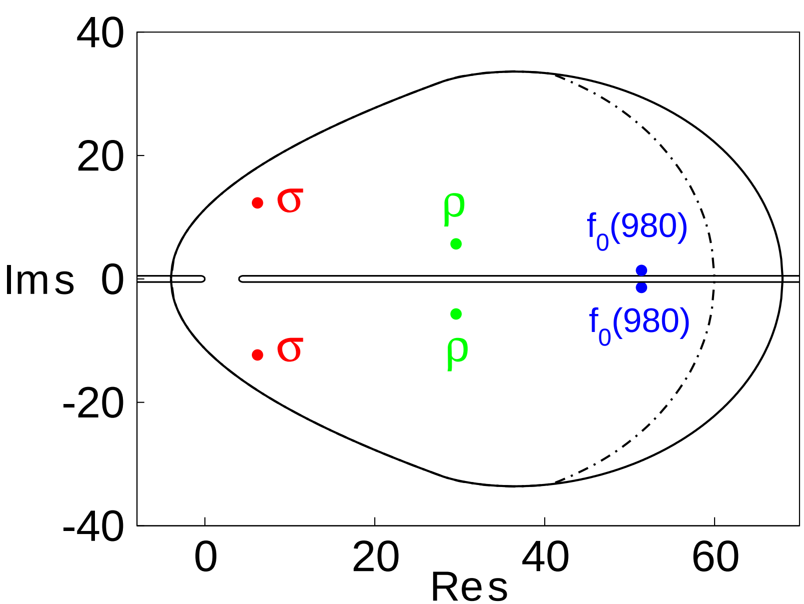
G. Mahoux, S. M. Roy & G. Wanders, Nucl. Phys. B 70 (1974) 297.

$\Rightarrow$ exact representation for $S_0^0(s)$ in this region
do not need to parametrize the amplitude

evaluation of the pole position

- insert our solutions of the Roy equations
  for the central solution, $S_0^0(s)$ has two pairs of zeros in the region of validity of the representation:

$$s = (6.2 \pm i\,12.3)\,M_\pi^2 \quad \sigma$$
$$s = (51.4 \pm i\,1.4)\,M_\pi^2 \quad f_0(980)$$



s in units of M_π^2

$\Rightarrow$ 1. lowest resonance of QCD has vacuum quantum numbers

2. pole on lower half of sheet II occurs in vicinity of

$$m_\sigma = 441 - i\,272\,\text{MeV}$$
$$= M_\sigma - \tfrac{i}{2}\Gamma_\sigma$$

error analysis

- results depend on phenomenological input used when solving the Roy equations, subject to uncertainties can follow error propagation explicitly
- pole position of $f_0(980)$ sensitive to input used for $\eta_0^0(s)$
- pole position of σ mainly depends on 3 input variables:

  $a_0^0, a_0^2, \delta_A \equiv \delta_0^0$(800 MeV)

  - information about a_0^0, a_0^2 is in good shape
  - substantial uncertainties in phenomenology of δ_A
  - use conservative range: $\delta_A = 82.3^\circ{}^{+10^\circ}_{-4^\circ}$

error analysis

- noise from remaining input variables is very small:

  $m_\sigma = (441 \pm 4) - i(272 \pm 6)$ MeV

  but the values of a_0^0, a_0^2, δ_A are crucial:

$$
\begin{aligned}
m_\sigma = &\,(441 \pm 4) - i(272 \pm 6) \\
&+ (-2.4 + i\,3.8)\,\frac{a_0^0 - 0.22}{0.005} \\
&+ (0.8 - i\,4.0)\,\frac{a_0^2 + 0.0444}{0.001} \\
&+ (5.3 + i\,3.3)\,\frac{\delta_A - 82.3}{3.4}
\end{aligned}
$$

numbers in MeV

- final result: insert the predictions for a_0^0, a_0^2, use the phenomenological range for δ_A and add errors up:

$$m_\sigma = 441\,{}^{+16}_{-8} \; - i\,272\,{}^{+9}_{-13} \text{ MeV}$$

curvature due to the left hand cut

- left hand cut generates curvature
  main contribution on the left stems from the ρ
- most pole determinations neglect the left hand cut
  pole from σ is too close for this to be justified
- can estimate contributions from left hand cut with χPT

  Z.Y. Zhou, G.Y. Qin, P. Zhang, Z.G. Xiao, H.Q. Zheng, N. Wu, JHEP 0502 (2005) 043

  estimate is crude $\Rightarrow$ sizeable uncertainties
  outcome for pole position agrees with our result

calculate pole position from phenomenology

- ignore the representation of the scattering amplitude obtained from the Roy equations
- instead use a phenomenological one

  J. R. Peláez & F. J. Ynduráin Phys. Rev. D71 (2005) 074016 $\longleftarrow$ PY

  (improved representation for energies above 1 GeV:

  R. Kaminski, J. R. Peláez & F. J. Ynduráin, hep-ph/0603170)
- insert it in formula for $S_0^0(s)$ and calculate the zeros with the central values of PY, this gives

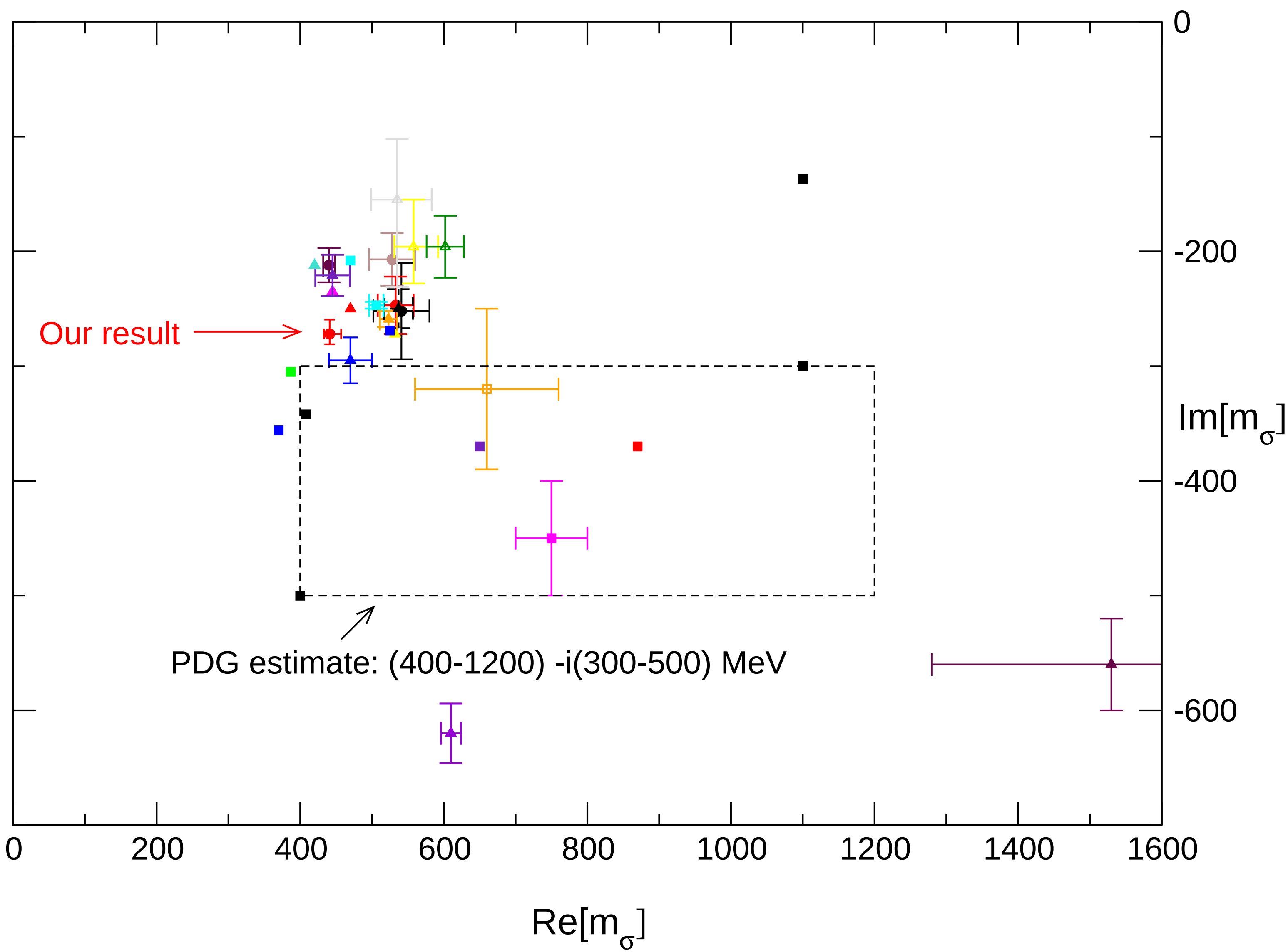
  $m_\sigma = 445 - i\,241$ MeV
- uncertainties in phenomenology are large

  those in a_0^0, a_0^2 alone give

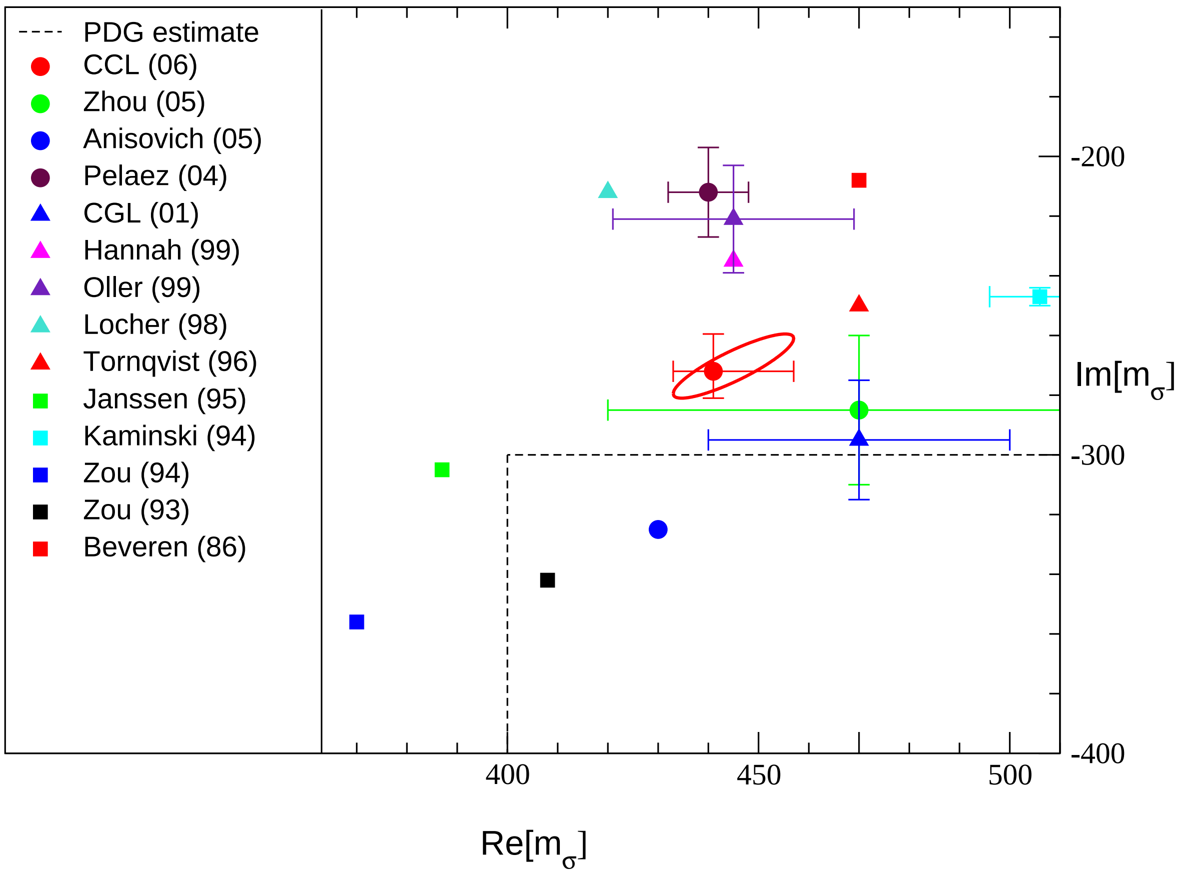
  $m_\sigma = (445 \pm 8) - i(241 \pm 22)$ MeV

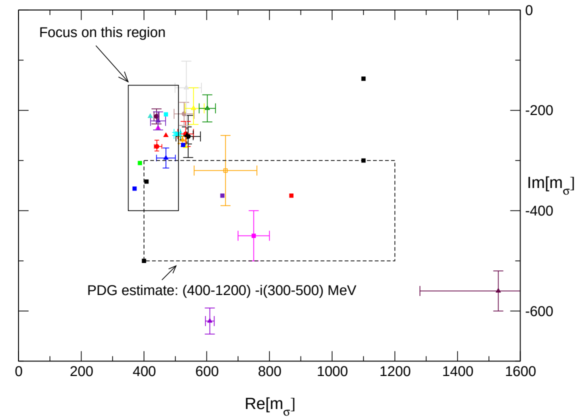
$\Rightarrow$ calculation confirms our result, but errors are larger

comparison with compilation of PDG



vicinity of the pole

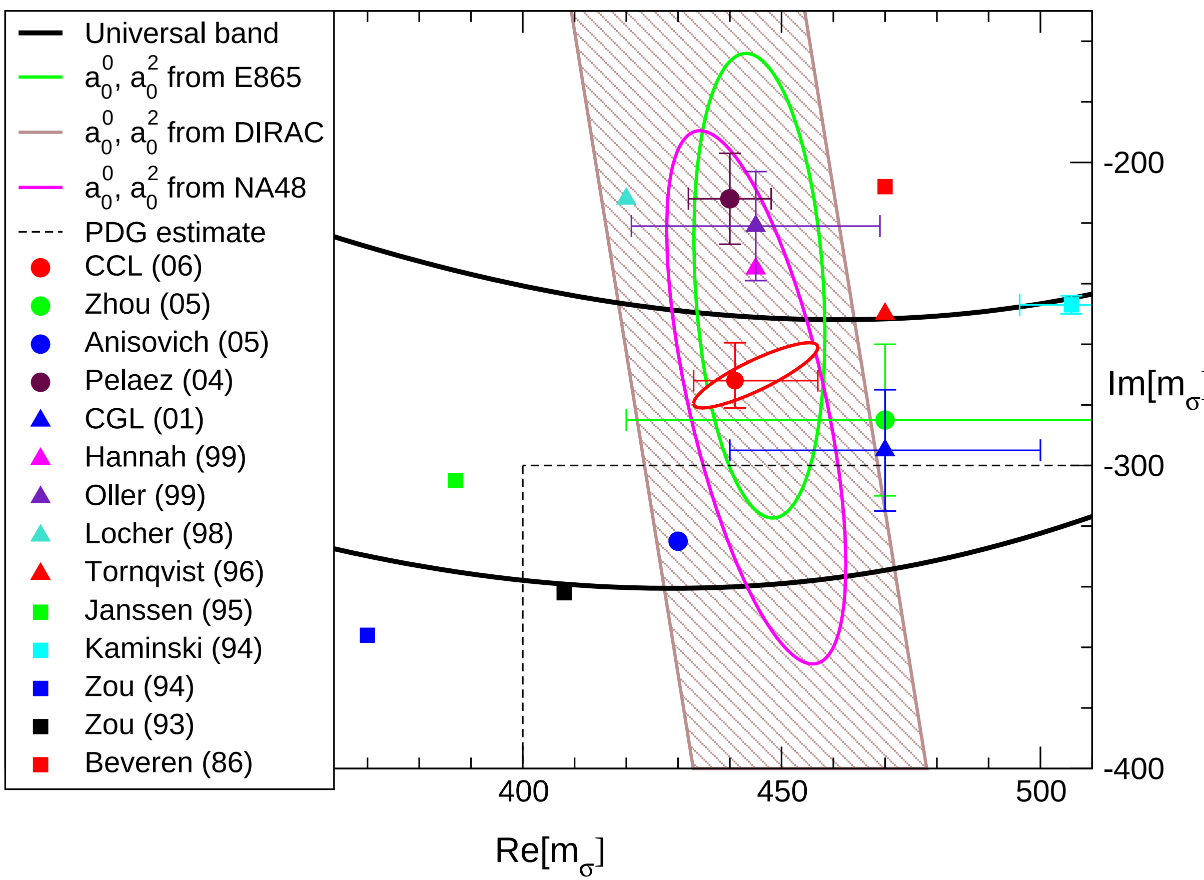




results for Re$[m_\sigma]$ and Im$[m_\sigma]$ are strongly correlated

ignore the theoretical predictions for a_0^0, a_0^2

- replace the low energy theorems for a_0^0, a_0^2 by the experimental results from E865, DIRAC and NA48
- $a_0^0, a_0^2 \in$ universal band



why are our errors so incredibly small ?

- the σ occurs at low energies
- at low energies, the subtraction term dominates

$$t_0^0(s) \simeq a_0^0 + (2a_0^0 - 5a_0^2)\frac{(s-4M_\pi^2)}{12M_\pi^2}$$

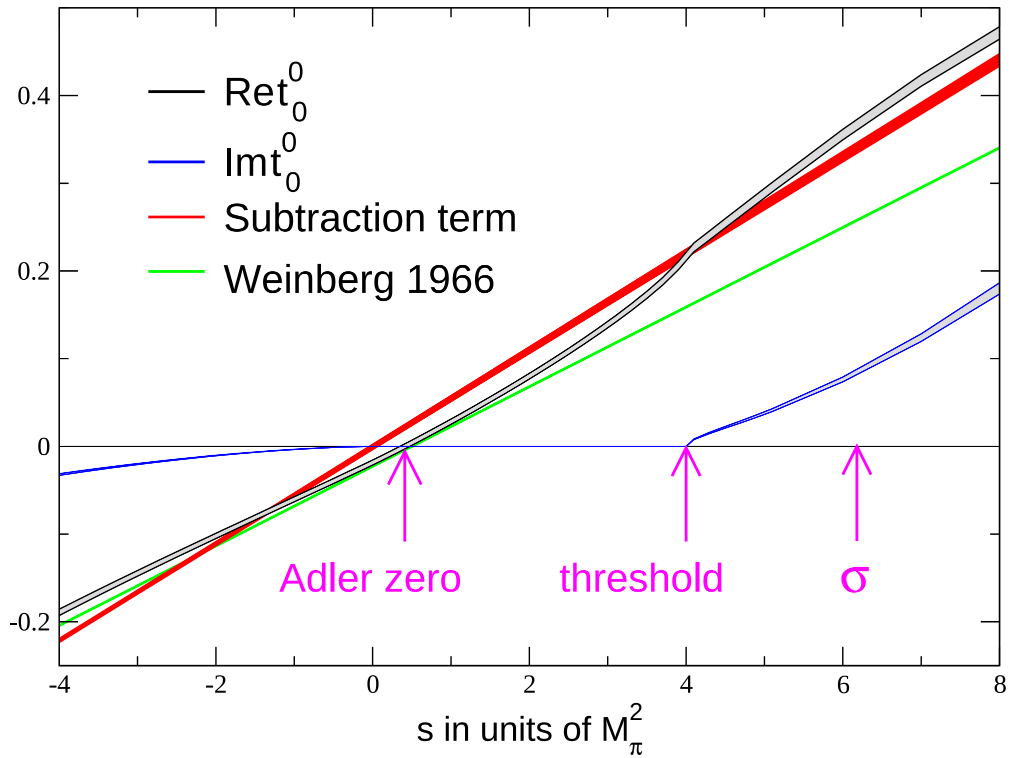
insert low energy theorem for a_0^0, a_0^2

$\Rightarrow$ Roy equation reduces to Weinberg formula

$$t_0^0(s) \simeq \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

dispersion integrals only represent a correction

at low energies, the subtraction term dominates



$$s = (0.41 \pm 0.06)\, M_\pi^2 \quad \text{Adler zero}$$

$$s = (6.2 - i\, 12.3)\, M_\pi^2 \quad \text{pole from } \sigma$$

at low energies, Goldstone bosons interact only weakly

estimate pole position on back of an envelope

- approximate $t_0^0(s)$ with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

where are the zeros of $S_0^0(s)$ in this approximation ?

$$1 + 2\,i\,\sqrt{1 - 4M_\pi^2/s}\;t_0^0(s) = 0$$

$\Rightarrow$ cubic equation for s

- pair of complex zeros, $m_\sigma = 365 - i\,291$ MeV
- correction from higher orders amounts to

$$\Delta m_\sigma = 76\,{}^{+16}_{-8}\; + i\,19\,{}^{+9}_{-13}\ \text{MeV}$$

for the quantity that counts, the accuracy is modest

- Real zero on sheet II, near $s = 0$ (full amplitude has kinematic singularity: vanishes on sheet II at $s = 0$)

physical interpretation of the σ

- the head of the dragon is not made of glue
- the dragon likes flavoured food, pions in particular

  Markushin & Locher 1999

- physics of the $\sigma \in$ Goldstone boson dynamics
  $\Rightarrow$ wave function has large tetra-quark component

  Jaffe 1977

- physics of the $f_0(980) \in$ Goldstone boson dynamics
  interaction among two kaons is relevant
- physics of the $\kappa \in$ Goldstone boson dynamics
  Roy-Steiner equations for $K\pi$ scattering

  Büttiker, Descotes-Genon & Moussallam 2006

physical interpretation of the κ

- oven fresh result from Roy-Steiner analysis:

  $$m_\kappa = (658 \pm 13) - i\,(278.5 \pm 12)\,\text{MeV}$$

  Descotes-Genon & Moussallam, hep-ph/0607133

- back-of-the-envelope calculation for $K\pi$ gives

  $$m_\kappa = 671 - i\,262\,\text{MeV}$$

$\Rightarrow$ physics of σ and κ is very similar

remark on $K\pi$ scattering

- 2 subtraction constants, dominate at low energies: $a_0^{\frac{1}{2}}$ (positive), $a_0^{\frac{3}{2}}$ (negative, small) $\leftrightarrow$ a_0^0, a_0^2 predictions less accurate: rely on expansion in m_s
- SU(2)×SU(2) theorem for $a_0^- = \frac{1}{3}(a_0^{\frac{1}{2}} - a_0^{\frac{3}{2}})$:

$$a_0^- = \frac{M_\pi^2}{8\pi F_\pi^2(1+M_\pi/M_K)}\{1+O(M_\pi^2)\}$$

$$\text{compare } \pi\pi : \quad a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\{1+O(M_\pi^2)\}$$

- final state interaction in $K\pi$ weaker than in $\pi\pi$

$\Rightarrow$ corrections for a_0^- should be even smaller than for a_0^0

- indeed, one loop correction in a_0^- is 12% [a_0^0: 25%]

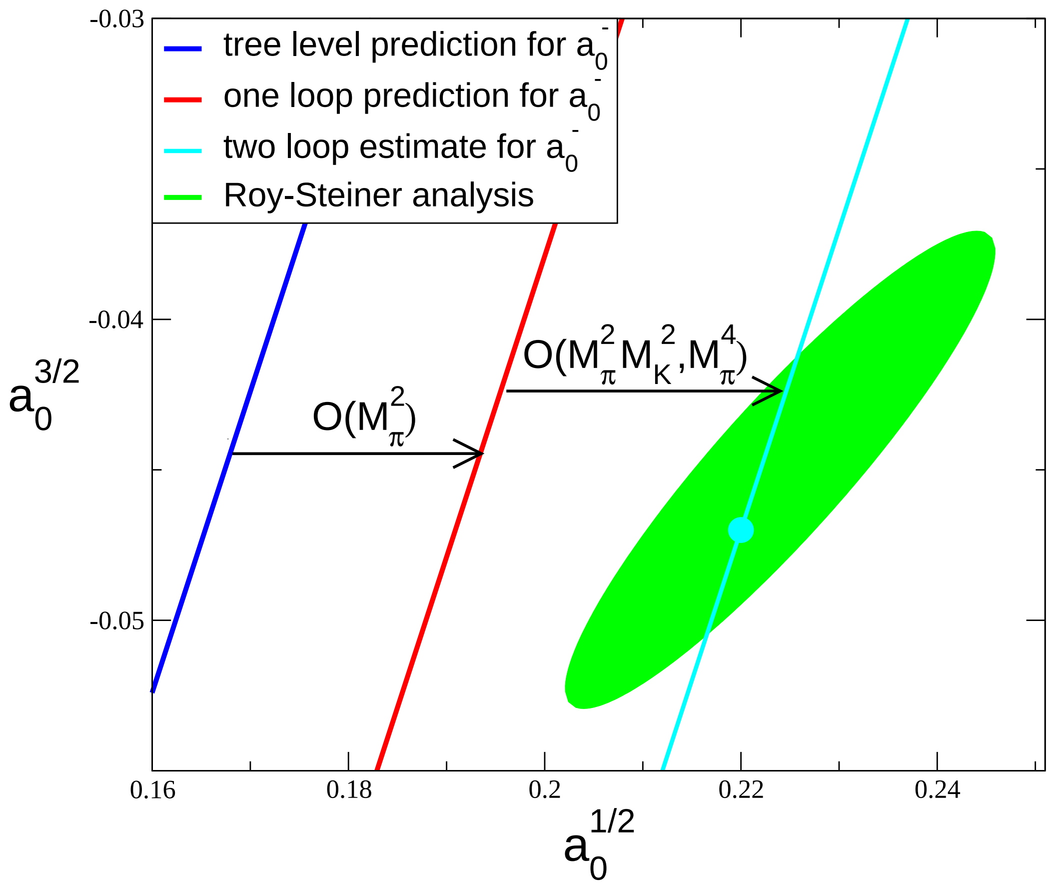
Roessl (1999), Kubis & Meissner (2002)

puzzle

- phenomenological analysis based on Roy-Steiner does not agree well with the one loop prediction for a_0^-

  Büttiker, Descotes-Genon & Moussallam 2004
- estimate for the $O(p^6)$ couplings gives large correction

  Bijnens, Dhonte & Talavera 2004, detailed analysis: Schweizer 2005



?

need to solve the puzzle

- does the expansion in powers of momenta fail already at threshold, because $M_K + M_\pi > 2M_\pi$?

$\Rightarrow$ if so, fix the subtractions at $s = u,\ t = 2M_\pi^2$

Cheng-Dashen point, compare Roy analysis of $\pi\pi$, Colangelo, Gasser & L. 2001

- resonance model of Bijnens et al. implies that terms of $O(M_\pi^2 M_K^2, M_\pi^4)$ are larger than terms of $O(M_\pi^2)$

$\Rightarrow$ looks supernatural – physics behind the phenomenon ?

- a_0^- can be measured by means of $K\pi$ atoms

  is there a reliable prediction and if so, what is it ?

conclusion

- low energy pion physics: theory ahead of experiment
  - precision experiments carried out and under way
  - lattice makes slow, but steady progress
  - so far, all tests confirm the theory
  - can extend χPT with dispersive methods
- limitations of the method:
  - calculations cannot be done on back of an envelope
  - method still only covers low energies
  - only a few applications have been worked out:
    $\pi\pi$ scattering, pion form factors, hadronic vacuum polarization in SM prediction for muon $g-2$
    $\gamma\gamma \to \pi^0\pi^0$ M. Pennington, hep-ph/0604212
- much is yet to be done: $J/\psi \to \omega\pi\pi$, $D \to 3\pi$, ... πK, κ, ...

conclusion

- model independent method for analytic continuation
  - the lowest resonance of QCD occurs at

    $$M_\sigma = 441\,{}^{+16}_{-8}\ \text{MeV} \qquad \Gamma_\sigma = 544\,{}^{+18}_{-25}\ \text{MeV}$$

    and carries vacuum quantum numbers
  - crossing symmetry plays an essential role:
    fixes contributions from left hand cut
    ensures fast convergence, low energy dominance
  - pole occurs at low value of s, closer to left hand cut than to singularities from $K\bar{K}$, $f_0(980)$
  - result for Γ_σ relies on theory for a_0^2
    experiments concerning a_0^2 would be most welcome

VISIT THE RED DRAGON

GENTLE ANIMAL

LOOK IN HIS EYES FROM CLOSE

SMELL HIS GOOD BREATH

BRING YOUR PIONS ALONG AND

FEED HIM YOURSELF

The management denies responsibility for incidents involving the dragon's tail