Introduction to chiral perturbation theory
II Higher orders, loops, applications

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Outline

The chiral Lagrangian to higher orders

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots$$

$\mathcal{L}_2$	contains	$(2,2)$	constants	
$\mathcal{L}_4$	contains	$(7,10)$	constants	Gasser, Leutwyler (84)
$\mathcal{L}_6$	contains	$(53,90)$	constants	Bijnens, GC, Ecker (99)

The number in parentheses are for an $SU(N)$ theory with $N = (2,3)$

The $\mathcal{L}_4$ Lagrangian

$$
\begin{aligned}
\mathcal{L}_4 \quad = \quad & L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
& + L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
& + L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + U^\dagger\chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
& + L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
& - iL_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \\
& + L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}
$$

$$
\begin{aligned}
D_\mu U \quad &= \quad \partial_\mu U - i r_\mu U + i U l_\mu \qquad \chi = 2B(s + ip) \\
F_R^{\mu\nu} \quad &= \quad \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \\
r_\mu \quad &= \quad v_\mu + a_\mu \qquad l_\mu = v_\mu - a_\mu
\end{aligned}
$$

Why go beyond $O(p^2)$? Why loops?

- Why not? Chiral Symmetry forbids $O(p^0)$ interactions between pions, but allows for all higher orders
- Unitarity requires that if an amplitude at order p^2 is purely real, at order p^4 its imaginary part is nonzero.
  Take the $\pi\pi$ scattering amplitude. The elastic unitarity relation for the partial waves t_ℓ^I of isospin I and angular momentum ℓ reads:

$$Im\, t_\ell^I = \sqrt{1 - \frac{4M_\pi^2}{s}} |t_\ell^I|^2 \qquad (1)$$

- The correct imaginary parts are generated automatically by loops
- The divergences occuring in the loops can be disposed of just like in a renormalizable field theory

Effective field theory

The method of effective quantum field theory provides a rigorous framework to compute Green functions that respect all the good properties we require: symmetry, analyticity, unitarity

The method yields a systematic expansion of the Green functions in powers of momenta and quark masses

In the following I will discuss in detail how this works when you consider loops:

- I will consider the finite, analytically nontrivial part of the loops and discuss in detail its physical meaning
- I will consider the divergent part of the loops and discuss how the renormalization program works

Scalar form factor of the pion

$$\langle \pi^i(p_1)\pi^j(p_2)|\hat{m}(\bar{u}u+\bar{d}d)|0\rangle =: \delta^{ij}\Gamma(t) \ , \qquad t=(p_1+p_2)^2 \ ,$$

At tree level:

$$\Gamma(t) = 2\hat{m}B = M_\pi^2 + O(p^4) \ ,$$

in agreement with the Feynman–Hellman theorem:
the expectation value of the perturbation in an eigenstate of the total Hamiltonian determines the derivative of the energy level with respect to the strength of the perturbation:

$$\hat{m}\frac{\partial M_\pi^2}{\partial \hat{m}} = \langle \pi|\hat{m}\bar{q}q|\pi\rangle = \Gamma(0) \ .$$

This matrix element is relevant for the decay $h \to \pi\pi$, which, for a light Higgs would have been the main decay mode

Donoghue, Gasser & Leutwyler (90)

Dispersion relation for $\Gamma(t)$

For $t \geq 4M_\pi^2$ $\mathrm{Im}\,\Gamma(t) \neq 0$. $\Gamma(t)$ is analytic everywhere else in the complex t plane, and obeys the following dispersion relation: $\bar{\Gamma}(t) = \Gamma(t)/\Gamma(0)$

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\frac{\mathrm{Im}\,\bar{\Gamma}(t')}{t'-t}$$

Unitarity implies $\qquad [\sigma(t) = \sqrt{1 - 4M_\pi^2/t}]$

$$\mathrm{Im}\,\bar{\Gamma}(t) = \sigma(t)\bar{\Gamma}(t){t_0^0}^*(t) = \bar{\Gamma}(t)e^{-i\delta_0^0}\sin\delta_0^0 = |\bar{\Gamma}(t)|\sin\delta_0^0$$

where t_0^0 is the S–wave, $I = 0$ $\pi\pi$ scattering amplitude

Strictly speaking, the above unitarity relation is valid only for $t \leq 16M_\pi^2$. To a good approximation, however, it holds up to the $K\bar{K}$ threshold

Dispersion relation and chiral counting

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{|\bar{\Gamma}(t')| \sin\delta_0^0(t')}{t'-t}$$

$$b \sim O(1)\left(1 + O(M_\pi^2)\right)$$

$$\delta_0^0 \sim O(p^2)\left(1 + O(p^2)\right)$$

There are two $O(p^2)$ correction to $\bar{\Gamma}$:

1. $O(1)$ contribution to b;
2. the dispersive integral containing the $O(p^2)$ phase δ_0^0.

Notice that the latter is fixed by unitarity and analyticity

Are these respected by the one loop calculation?

Bierfrage: Beweis?

Hints:

- Subtract $\bar{J}(t)$ once more

$$\bar{J}(t) = \frac{t}{96\pi^2} + \frac{t^2}{16\pi^2} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\sigma(t')}{t' - t}$$

- Trick to pull out a linear term from the dispersive integral:

$$\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{t'\sigma(t')}{t' - t} = t \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\sigma(t')}{t' - t} + \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \sigma(t')$$

Renormalization at one loop

$$\int \frac{d^4 l}{(2\pi)^4} \frac{\{p^2, p\cdot l, l^2\}}{(l^2 - M^2)((p-l)^2 - M^2)} \;, \qquad p = p_1 + p_2$$

$$\sim \underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2)}}_{T(M^2)} + p^2 \underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2)((p-l)^2 - M^2)}}_{J(p^2)}$$

$$T(M^2) = a + bM^2 + \bar{T}(M^2) \qquad J(t) = J(0) + \bar{J}(t)$$

$\bar{T}(M^2)$ and $\bar{J}(t)$ are finite

$$\Gamma(t) \sim M^2 \left[1 + \underbrace{bM^2 + tJ(0)}_{\text{divergent part}} + \bar{T}(M^2) + \bar{J}(t) \right]$$

Counterterms

$$\mathcal{L}_2 \quad \Rightarrow \quad \Gamma^{(2)}(t) \sim M^2$$

$$\mathcal{L}_4 \quad \Rightarrow \quad \Gamma^{(4)}(t) \sim l_3 M^4 + l_4 M^2 t$$

To remove the divergences one only needs to properly define the couplings ($l_{3,4}$) in the lagrangian at order $O(p^4)$

Quote from Weinberg's book on QFT, vol. I: "(...) as long as we include every one of the infinite number of interactions allowed by symmetries, the so–called non–renormalizable theories are actually just as renormalizable as renormalizable theories."

Chiral logarithms

Scalar radius of the pion

$$\Gamma(t) = \Gamma(0)\left[1 + \frac{1}{6}\langle r^2\rangle_S^\pi t + O(t^2)\right]$$

$$\langle r^2\rangle_S^\pi \sim J(0) = \int \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2 - M^2)^2} \sim \ln\frac{M^2}{\Lambda^2}$$

The integral is UV divergent, but also IR divergent if $M \to 0$:

$$\lim_{M^2\to 0}\langle r^2\rangle_S^\pi \sim \ln M^2 \;\;,$$

The extension of the cloud of pions surrounding a pion (or any other hadron) goes to infinity if pions become massless (Li and Pagels '72)

Chiral perturbation theory

- Chiral perturbation theory provides a rigorous framework to compute Green functions that respect all the good properties we require:
  symmetry, analyticity, unitarity
- The method yields a systematic expansion of the Green functions in powers of momenta and quark masses
- The method has been rigorously established and can be formulated as a set of calculational rules:

LO	tree level diagrams with $\mathcal{L}_2$
NLO	tree level diagrams with $\mathcal{L}_4$
	1-loop diagrams with $\mathcal{L}_2$
NNLO	tree level diagrams with $\mathcal{L}_6$
	2-loop diagrams with $\mathcal{L}_2$
	1-loop diagrams with one vertex from $\mathcal{L}_4$

Chiral symmetry and renormalization

To remove the divergent part in $\Gamma(t)$ we have to fix the divergent part of chiral–invariant operator of order $O(p^4)$

e.g. $$\langle D_\mu U^\dagger D^\mu U\rangle\langle B\mathcal{M}(U+U^\dagger)\rangle \sim \ldots + M^2\phi^2\partial_\mu\phi^4\partial^\mu\phi^6 + \ldots$$

1. Do we have a proof that quantum effects do not introduce violations of the chiral symmetry? Or that one can build a chiral invariant generating functional only with a path integral over a chiral invariant classical action?
2. Is there a tool that allows one to calculate the divergences keeping chiral invariance explicit in every step of the calculation?

Generating functional

- Consider a system with a spontaneously broken symmetry G. Define the generating functional as:

$$e^{iZ\{f\}} = \sum_{n=0} \frac{i^n}{n!} \int dx_1 \dots dx_n f^{i_1}_{\mu_1} \dots f^{i_n}_{\mu_n} \langle 0 | T J^{\mu_1}_{i_1} \dots J^{\mu_n}_{i_n} | 0 \rangle \quad ,$$

- The generating functional is invariant under gauge transformations of the external fields:

$$Z\{T(g)f\} = Z\{f\} \quad ,$$

where:

$$T(g)f_\mu = D(g_x) f_\mu(x) D^{-1}(g_x) - i\partial_\mu D(g_x) D^{-1}(g_x)$$

Leutwyler's theorem

What is the most general way of constructing a chiral-invariant generating functional out of a path integral over the Goldstone boson degrees of freedom?

For Lorentz–invariant theories in 4 dimensions, a path integral constructed with gauge–invariant lagrangians is a necessary and sufficient condition to obtain a gauge–invariant generating functional

The theorem also includes the case in which the symmetry is anomalous and the case in which the symmetry is explicitly broken

Chiral invariant renormalization

- Gasser & Leutwyler (84) have shown that, using the background field method and heat kernel techniques, the calculation of the divergences at one loop – and the corresponding renirmalization – can be performed in an explicitly chiral invariant manner
- The method has been extended and applied to two loops (Bijnens, GC & Ecker 98). After a long and tedious calculation, the divergent parts of all the counterterms at $\mathcal{O}(p^6)$ has been provided
- The renormalization of CHPT up to two loops has been performed explicitly: the calculation of any amplitude at two loops can be immediately checked by comparing the divergent part of Feynman diagrams to the divergent parts of the relevant counterterms

$\pi\pi$ scattering at NLO

$$
\begin{aligned}
a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1+\frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi+\frac{200\pi F_\pi^2 M_\pi^2}{7}(a_2^0+2a_2^2)\right. \\
&- \left.\frac{M_\pi^2}{672\pi^2 F_\pi^2}(15\bar{l}_3-353)\right] = 0.16\cdot 1.25 = 0.20 \\
2a_0^0-5a_0^2 &= \frac{3M_\pi^2}{4\pi F_\pi^2}\left[1+\frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi+\frac{41M_\pi^2}{192\pi^2 F_\pi^2}\right] = 0.624
\end{aligned}
$$

Gasser and Leutwyler (83)

$\pi\pi$ scattering at NLO

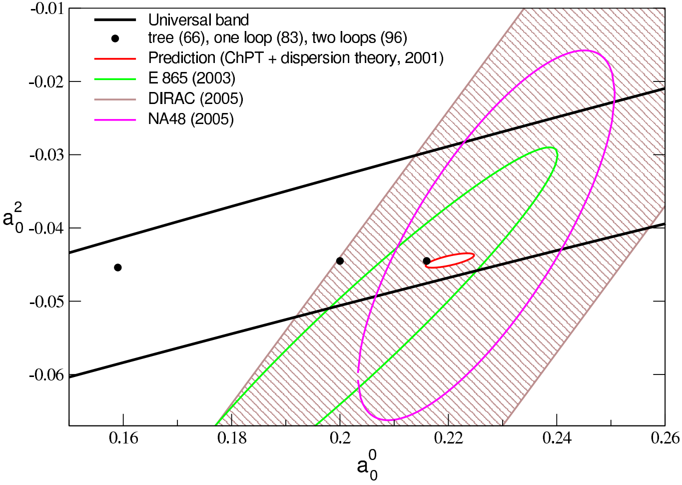
$$
\begin{aligned}
a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi + \frac{200\pi F_\pi^2 M_\pi^2}{7}(a_2^0 + 2a_2^2)\right. \\
&\quad \left. - \frac{M_\pi^2}{672\pi^2 F_\pi^2}(15\bar{l}_3 - 353)\right] = 0.16 \cdot 1.25 = 0.20 \\
a_0^0 - a_0^2 &= 0.245
\end{aligned}
$$

Gasser and Leutwyler (83)

$a_0^0 = 0.26 \pm 0.05$	Rosselet et al. (77)
$a_0^0 = 0.216 \pm 0.013 \pm 0.003$	Pislak et al. (E865) (03)
$\lvert a_0^0 - a_0^2\rvert = 0.264\,^{+0.033}_{-0.020}$	Adeva et al. (DIRAC) (05)
$a_0^0 - a_0^2 = 0.268 \pm 0.010 \pm 0.013$	Batley et al. (NA48/2) (06)

Comparison of NNLO prediction and data $\Rightarrow$ talk of Leutwyler

$\pi\pi$ scattering at NLO



K_{l3} decays at NLO

$$
\begin{aligned}
\langle K^+|\bar{u}\gamma_\mu s|\pi^0\rangle &= \frac{1}{\sqrt{2}}\left[(p'+p)_\mu f_+(t)+(p'-p)_\mu f_-(t)\right] \\
f_{+,0}(t) &= f_{+,0}(0)\left(1+\lambda_{+,0}\frac{t}{M_\pi^2}+\dots\right) \\
f_0 &= f_+ + \frac{t}{M_K^2-M_\pi^2}f_- \\
\lambda_+ &= \frac{M_\pi^2}{6}\langle r\rangle_V^\pi + \Delta_+ = 0.031 \\
\lambda_0 &= \frac{M_\pi^2}{M_K^2-M_\pi^2}\left(\frac{F_K}{F_\pi}-1\right)+\Delta_0 = 0.017
\end{aligned}
$$

Gasser and Leutwyler (85)

K_{l3} decays at NLO

$$\lambda_+ = \frac{M_\pi^2}{6}\langle r\rangle_V^\pi + \Delta_+ = 0.031$$
$$\lambda_0 = \frac{M_\pi^2}{M_K^2 - M_\pi^2}\left(\frac{F_K}{F_\pi} - 1\right) + \Delta_0 = 0.017$$

Gasser and Leutwyler (85)

Experimental values:

Exp.	$10^3\lambda_+$	$10^3\lambda_0$
ISTRA ($K^-_{\mu 3}$)	29.7 ± 1.6	19.6 ± 1.4
ISTRA (K^-_{e3})	24.7 ± 1.6	
KTeV ($K_{L\ e,\mu 3}$)	20.6 ± 1.8	13.7 ± 1.3
NA48/2 ($K_{L\ e3}$)	28.0 ± 1.9	
NA48/2 ($K_{L\ \mu 3}$)	26.0 ± 1.2	12.0 ± 1.7
KLOE ($K_{L\ e3}$)	25.5 ± 1.5	

K_{l3} decays at NLO

$$\lambda_+ \quad = \quad \frac{M_\pi^2}{6}\langle r\rangle_V^\pi + \Delta_+ = 0.031$$

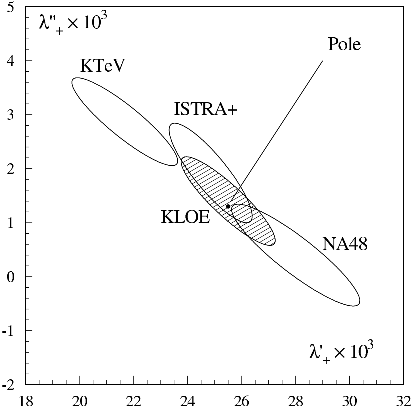


Figure by KLOE, hep-ex/0601038

K_{l3} decays at NNLO

- K_{l3} amplitude known at NNLO

  Post & Schilcher (02)

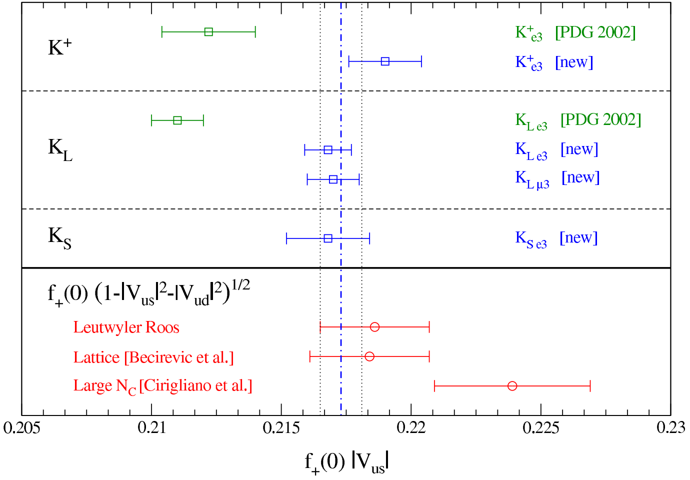
  Bijnens & Talavera (03)

- Interesting relation among $f_+(0)$, slope and curvature

$$
\begin{aligned}
\tilde{f}_0(t) &:= f_0(t) + \frac{t}{M_K^2 - M_\pi^2}(1 - F_K/F_\pi) \\
\tilde{f}_0(t) &= 1 - \frac{8}{F_\pi^4}(C_{12}^r + C_{34}^r)(M_K^2 - M_\pi^2)^2 \\
&+ \frac{8t}{F_\pi^4}(2C_{12}^r + C_{34}^r)(M_K^2 + M_\pi^2) - \frac{8t^2}{F_\pi^4}C_{12}^r + \Delta(t)
\end{aligned}
$$

- The value of $f_+(0)$ can be predicted in terms of measured quantities $\Rightarrow$ extraction of V_{us} from data on K_{e3}

K_{l3} decays at NNLO



From Blucher et al. CKM05-WG1, hep-ph/0512039

Areas of application

- Pions, kaons and etas
  - Purely strong interactions ((semi)leptonic decays)
  - Weak nonleptonic (radiative) decays
  - Electromagnetic interactions
  - Decays of electromagnetically bound states

Areas of application

- Pions, kaons and etas
- Nucleons
  - One nucleon sector: πN or KN scattering
  - Electromagnetic interactions
  - Two nucleon sector: NN scattering, nuclear forces

Areas of application

- Pions, kaons and etas
- Nucleons
- Connections to lattice QCD
  - (Partially) Quenched Chiral Perturbation Theory
  - Study of finite–volume and –temperature effects
  - Extrapolation to the chiral limit
  - Extrapolation to zero lattice spacing

Areas of application

- Pions, kaons and etas
- Nucleons
- Connections to lattice QCD
- Condensed–matter systems with spontaneous symmetry breaking
  - Ferromagnets
  - Antiferromagnets in $d = 3$ or $d = 2$

Areas of application

- Pions, kaons and etas
- Nucleons
- Connections to lattice QCD
- Condensed–matter systems with spontaneous symmetry breaking
- Electroweak symmetry breaking – models in which the electroweak symmetry is broken strongly

Areas of application

- Pions, kaons and etas
- Nucleons
- Connections to lattice QCD
- Condensed–matter systems with spontaneous symmetry breaking
- Electroweak symmetry breaking – models in which the electroweak symmetry is broken strongly
- General relativity as an effective field theory

Summary

- The finite, analytically nontrivial part of the one loop integrals automatically generates the correct imaginary parts, as required by unitarity.
- Effective quantum field theory is a systematic method to generate a perturbative solution of dispersion relations
- The UV divergences encountered in loop integrals can be removed according to standard renormalization methods
- Some loop integrals have also an IR singular behaviour which has a very clear physical meaning, and again shows the necessity of taking loop effects into account
- Leutwyler's theorem: doing a path integral over an effective Lagrangian is the most general way to construct an invariant generating functional
- I have illustrated the method discussing two applications:
  - the $\pi\pi$ S-wave scattering lengths
  - K_{e3} decays and the extraction of V_{us}