Introduction to chiral perturbation theory
II Higher orders, loops, applications

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Outline

The chiral Lagrangian to higher orders

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots$$

$\mathcal{L}_2$	contains	$(2, 2)$	constants	
$\mathcal{L}_4$	contains	$(7, 10)$	constants	Gasser, Leutwyler (84)
$\mathcal{L}_6$	contains	$(53, 90)$	constants	Bijnens, GC, Ecker (99)

The number in parentheses are for an $SU(N)$ theory with $N = (2, 3)$

The $\mathcal{L}_4$ Lagrangian

$$
\begin{aligned}
\mathcal{L}_4 \;=\;& L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
&+L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&-iL_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \\
&+L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}
$$

$$
\begin{aligned}
D_\mu U &= \partial_\mu U - i r_\mu U + i U l_\mu \qquad \chi = 2B(s+ip) \\
F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \\
r_\mu &= v_\mu + a_\mu \qquad l_\mu = v_\mu - a_\mu
\end{aligned}
$$

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- Unitarity requires that if an amplitude at order p^2 is purely real, at order p^4 its imaginary part is nonzero.
  Take the $\pi\pi$ scattering amplitude. The elastic unitarity relation for the partial waves t_ℓ^I of isospin I and angular momentum ℓ reads:

$$Im\, t_\ell^I = \sqrt{1 - \frac{4M_\pi^2}{s}}|t_\ell^I|^2 \tag{1}$$

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- The divergences occuring in the loops can be disposed of just like in a renormalizable field theory

Effective field theory

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- I will consider the finite, analytically nontrivial part of the loops and discuss in detail its physical meaning
- I will consider the divergent part of the loops and discuss how the renormalization program works

Scalar form factor of the pion

$$\langle \pi^i(p_1)\pi^j(p_2)|\hat{m}(\bar{u}u + \bar{d}d)|0\rangle =: \delta^{ij}\Gamma(t) \;, \qquad t = (p_1 + p_2)^2 \;,$$

At tree level:

$$\Gamma(t) = 2\hat{m}B = M_\pi^2 + O(p^4) \;,$$

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in agreement with the Feynman–Hellman theorem:
the expectation value of the perturbation in an eigenstate of the total Hamiltonian determines the derivative of the energy level with respect to the strength of the perturbation:

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This matrix element is relevant for the decay $h \to \pi\pi$, which, for a light Higgs would have been the main decay mode

Donoghue, Gasser & Leutwyler (90)

Dispersion relation for $\Gamma(t)$

For $t \geq 4M_\pi^2$ $\text{Im}\,\Gamma(t) \neq 0$. $\Gamma(t)$ is analytic everywhere else in the complex t plane, and obeys the following dispersion relation: $\bar{\Gamma}(t) = \Gamma(t)/\Gamma(0)$

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\text{Im}\,\bar{\Gamma}(t')}{t' - t}$$

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Unitarity implies $[\sigma(t) = \sqrt{1 - 4M_\pi^2/t}]$

$$\mathrm{Im}\,\bar{\Gamma}(t) = \sigma(t)\bar{\Gamma}(t)t_0^{0*}(t) = \bar{\Gamma}(t)e^{-i\delta_0^0}\sin\delta_0^0 = |\bar{\Gamma}(t)|\sin\delta_0^0$$

where t_0^0 is the S–wave, $I = 0$ $\pi\pi$ scattering amplitude

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Strictly speaking, the above unitarity relation is valid only for $t \leq 16M_\pi^2$. To a good approximation, however, it holds up to the $K\bar{K}$ threshold

Dispersion relation and chiral counting

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{|\bar{\Gamma}(t')| \sin \delta_0^0(t')}{t' - t}$$

$$b \sim O(1) \left(1 + O(M_\pi^2)\right)$$

$$\delta_0^0 \sim O(p^2) \left(1 + O(p^2)\right)$$

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There are two $O(p^2)$ correction to $\bar{\Gamma}$:

1. $O(1)$ contribution to b;
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Notice that the latter is fixed by unitarity and analyticity

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Are these respected by the one loop calculation?

Dispersion relation and one–loop CHPT

The full one–loop expression of $\bar{\Gamma}(t)$ reads as follows:

$$\bar{\Gamma}(t) = 1 + \frac{t}{16\pi^2 F_\pi^2}(\bar{l}_4 - 1) + \frac{2t - M_\pi^2}{2F_\pi^2}\bar{J}(t)$$

$$\bar{J}(t) = \frac{1}{16\pi^2}\left[\sigma(t)\ln\frac{\sigma(t)-1}{\sigma(t)+1} + 2\right]$$

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To prove that unitarity and analyticity are respected at this order is sufficient to add:

$$\delta_0^0(t) = \sigma(t)\frac{2t - M_\pi^2}{32\pi F_\pi^2} + O(p^4) \qquad \bar{J}(t) = \frac{t}{16\pi^2}\int_{4M_\pi^2}^{\infty}\frac{dt'}{t'}\frac{\sigma(t')}{t'-t}$$

Bierfrage: Beweis?

Hints:

- Subtract $\bar{J}(t)$ once more

$$\bar{J}(t) = \frac{t}{96\pi^2} + \frac{t^2}{16\pi^2} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\sigma(t')}{t' - t}$$

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- Trick to pull out a linear term from the dispersive integral:

$$\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{t'\sigma(t')}{t' - t} = t \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\sigma(t')}{t' - t} + \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \sigma(t')$$

Renormalization at one loop

$$\int \frac{d^4l}{(2\pi)^4} \frac{\{p^2, p\cdot l, l^2\}}{(l^2 - M^2)((p-l)^2 - M^2)} \,, \qquad p = p_1 + p_2$$

$$\sim \underbrace{\int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - M^2)}}_{T(M^2)} + p^2 \underbrace{\int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - M^2)((p-l)^2 - M^2)}}_{J(p^2)}$$

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$\bar{T}(M^2)$ and $\bar{J}(t)$ are finite

$$\Gamma(t) \sim M^2 \left[1 + \underbrace{bM^2 + tJ(0)}_{\text{divergent part}} + \bar{T}(M^2) + \bar{J}(t)\right]$$

Counterterms

$$\mathcal{L}_2 \quad \Rightarrow \quad \Gamma^{(2)}(t) \sim M^2$$

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To remove the divergences one only needs to properly define the couplings ($l_{3,4}$) in the lagrangian at order $O(p^4)$

Quote from Weinberg's book on QFT, vol. I: "(...) as long as we include every one of the infinite number of interactions allowed by symmetries, the so–called non–renormalizable theories are actually just as renormalizable as renormalizable theories."

Chiral logarithms

Scalar radius of the pion

$$\Gamma(t) = \Gamma(0)\left[1 + \frac{1}{6}\langle r^2\rangle_S^\pi t + O(t^2)\right]$$

$$\langle r^2\rangle_S^\pi \sim J(0) = \int \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2 - M^2)^2} \sim \ln\frac{M^2}{\Lambda^2}$$

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The extension of the cloud of pions surrounding a pion (or any other hadron) goes to infinity if pions become massless (Li and Pagels '72)

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  NLO tree level diagrams with $\mathcal{L}_4$
  1-loop diagrams with $\mathcal{L}_2$

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LO	tree level diagrams with $\mathcal{L}_2$
NLO	tree level diagrams with $\mathcal{L}_4$
	1-loop diagrams with $\mathcal{L}_2$
NNLO	tree level diagrams with $\mathcal{L}_6$
	2-loop diagrams with $\mathcal{L}_2$
	1-loop diagrams with one vertex from $\mathcal{L}_4$

Chiral symmetry and renormalization

To remove the divergent part in $\Gamma(t)$ we have to fix the divergent part of chiral–invariant operator of order $O(p^4)$

e.g. $$\langle D_\mu U^\dagger D^\mu U\rangle\langle B\mathcal{M}(U+U^\dagger)\rangle \sim \ldots + M^2\phi^2\partial_\mu\phi^4\partial^\mu\phi^6 + \ldots$$

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Chiral symmetry implies that after calculating the divergent part of $\Gamma(s)$ I also know the divergent part of the $6\pi \to 6\pi$ scattering amplitude

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1. Do we have a proof that quantum effects do not introduce violations of the chiral symmetry? Or that one can build a chiral invariant generating functional only with a path integral over a chiral invariant classical action?
2. Is there a tool that allows one to calculate the divergences keeping chiral invariance explicit in every step of the calculation?

Generating functional

- Consider a system with a spontaneously broken symmetry G. Define the generating functional as:

$$e^{iZ\{f\}} = \sum_{n=0} \frac{i^n}{n!} \int dx_1 \ldots dx_n f^{i_1}_{\mu_1} \ldots f^{i_n}_{\mu_n} \langle 0 | T J^{\mu_1}_{i_1} \ldots J^{\mu_n}_{i_n} | 0 \rangle \quad ,$$

where J^i_μ are the Noether's currents associated to the spontaneously broken symmetry G of the system, and f^μ_i external fields coupled to them

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- The generating functional is invariant under gauge transformations of the external fields:

$$Z\{T(g)f\} = Z\{f\} \quad ,$$

where:

$$T(g)f_\mu = D(g_x)f_\mu(x)D^{-1}(g_x) - i\partial_\mu D(g_x)D^{-1}(g_x)$$

Leutwyler's theorem

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For Lorentz–invariant theories in 4 dimensions, a path integral constructed with gauge–invariant lagrangians is a necessary and sufficient condition to obtain a gauge–invariant generating functional

The theorem also includes the case in which the symmetry is anomalous and the case in which the symmetry is explicitly broken

Chiral invariant renormalization

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- The method has been extended and applied to two loops (Bijnens, GC & Ecker 98). After a long and tedious calculation, the divergent parts of all the counterterms at $\mathcal{O}(p^6)$ has been provided
- The renormalization of CHPT up to two loops has been performed explicitly: the calculation of any amplitude at two loops can be immediately checked by comparing the divergent part of Feynman diagrams to the divergent parts of the relevant counterterms

$\pi\pi$ scattering at NLO

$$\begin{aligned} a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1+\frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi+\frac{200\pi F_\pi^2 M_\pi^2}{7}(a_2^0+2a_2^2)\right. \\ &- \left.\frac{M_\pi^2}{672\pi^2 F_\pi^2}(15\bar{l}_3-353)\right] = 0.16\cdot 1.25 = 0.20 \\ 2a_0^0-5a_0^2 &= \frac{3M_\pi^2}{4\pi F_\pi^2}\left[1+\frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi+\frac{41M_\pi^2}{192\pi^2 F_\pi^2}\right] = 0.624 \end{aligned}$$

Gasser and Leutwyler (83)

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&\quad \left. - \frac{M_\pi^2}{672\pi^2 F_\pi^2}(15\bar{l}_3 - 353)\right] = 0.16 \cdot 1.25 = 0.20 \\
a_0^0 - a_0^2 &= 0.245
\end{aligned}
$$

Gasser and Leutwyler (83)

$$a_0^0 = 0.26 \pm 0.05$$ Rosselet et al. (77)

$$a_0^0 = 0.216 \pm 0.013 \pm 0.003$$ Pislak et al. (E865) (03)

$$|a_0^0 - a_0^2| = 0.264 \ {}^{+0.033}_{-0.020}$$ Adeva et al. (DIRAC) (05)

$$a_0^0 - a_0^2 = 0.268 \pm 0.010 \pm 0.013$$ Batley et al. (NA48/2) (06)

$\pi\pi$ scattering at NLO

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi + \frac{200\pi F_\pi^2 M_\pi^2}{7}(a_2^0 + 2a_2^2) - \frac{M_\pi^2}{672\pi^2 F_\pi^2}(15\bar{l}_3 - 353)\right] = 0.16 \cdot 1.25 = 0.20$$

$$a_0^0 - a_0^2 = 0.245$$

Gasser and Leutwyler (83)

$$a_0^0 = 0.26 \pm 0.05$$ Rosselet et al. (77)

$$a_0^0 = 0.216 \pm 0.013 \pm 0.003$$ Pislak et al. (E865) (03)

$$|a_0^0 - a_0^2| = 0.264\,^{+0.033}_{-0.020}$$ Adeva et al. (DIRAC) (05)

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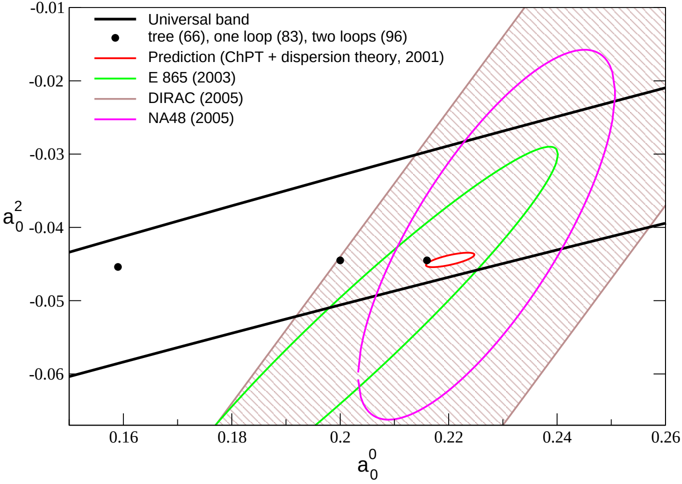
Comparison of NNLO prediction and data $\Rightarrow$ talk of Leutwyler

$\pi\pi$ scattering at NLO



K_{l3} decays at NLO

$$
\begin{aligned}
\langle K^+|\bar{u}\gamma_\mu s|\pi^0\rangle &= \frac{1}{\sqrt{2}}\left[(p'+p)_\mu f_+(t) + (p'-p)_\mu f_-(t)\right] \\
f_{+,0}(t) &= f_{+,0}(0)\left(1+\lambda_{+,0}\frac{t}{M_\pi^2}+\ldots\right) \\
f_0 &= f_+ + \frac{t}{M_K^2-M_\pi^2} f_- \\
\lambda_+ &= \frac{M_\pi^2}{6}\langle r\rangle_V^\pi + \Delta_+ = 0.031 \\
\lambda_0 &= \frac{M_\pi^2}{M_K^2-M_\pi^2}\left(\frac{F_K}{F_\pi}-1\right) + \Delta_0 = 0.017
\end{aligned}
$$

Gasser and Leutwyler (85)

K_{l3} decays at NLO

$$\lambda_+ = \frac{M_\pi^2}{6}\langle r\rangle_V^\pi + \Delta_+ = 0.031$$

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Experimental values:

Exp.	$10^3\lambda_+$	$10^3\lambda_0$
ISTRA ($K^-_{\mu 3}$)	29.7 ± 1.6	19.6 ± 1.4
ISTRA (K^-_{e3})	24.7 ± 1.6	
KTeV ($K_{L\ e,\mu 3}$)	20.6 ± 1.8	13.7 ± 1.3
NA48/2 ($K_{L\ e3}$)	28.0 ± 1.9	
NA48/2 ($K_{L\ \mu 3}$)	26.0 ± 1.2	12.0 ± 1.7
KLOE ($K_{L\ e3}$)	25.5 ± 1.5	

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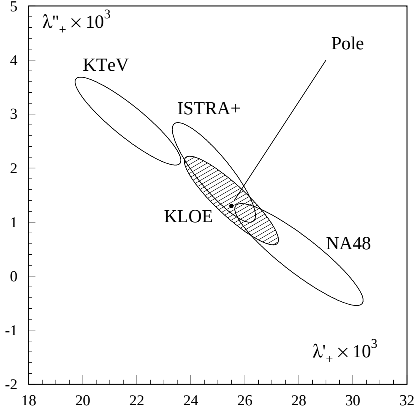

Figure by KLOE, hep-ex/0601038

K_{l3} decays at NNLO

- K_{l3} amplitude known at NNLO

Post & Schilcher (02)

Bijnens & Talavera (03)

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- Interesting relation among $f_+(0)$, slope and curvature

$$\tilde{f}_0(t) := f_0(t) + \frac{t}{M_K^2 - M_\pi^2}(1 - F_K/F_\pi)$$

$$\tilde{f}_0(t) = 1 - \frac{8}{F_\pi^4}(C_{12}^r + C_{34}^r)(M_K^2 - M_\pi^2)^2 + \frac{8t}{F_\pi^4}(2C_{12}^r + C_{34}^r)(M_K^2 + M_\pi^2) - \frac{8t^2}{F_\pi^4}C_{12}^r + \Delta(t)$$

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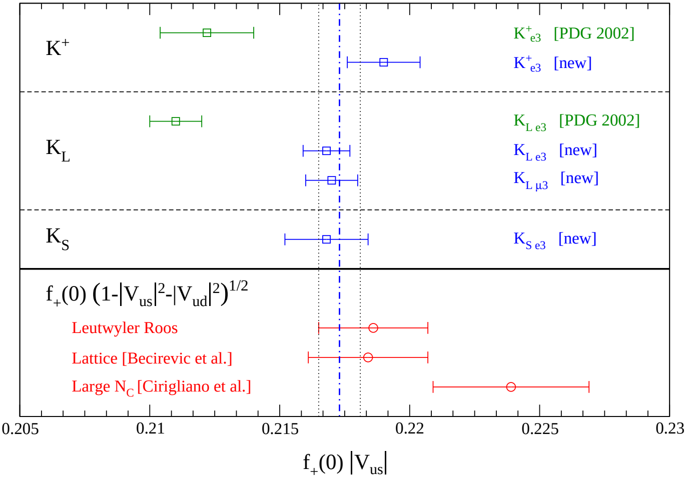
- The value of $f_+(0)$ can be predicted in terms of measured quantities $\Rightarrow$ extraction of V_{us} from data on K_{e3}

K_{l3} decays at NNLO



From Blucher et al. CKM05-WG1, hep-ph/0512039

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  - Two nucleon sector: NN scattering, nuclear forces

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- I have illustrated the method discussing two applications:
  - the $\pi\pi$ S-wave scattering lengths
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