Introduction to chiral perturbation theory
I Foundations

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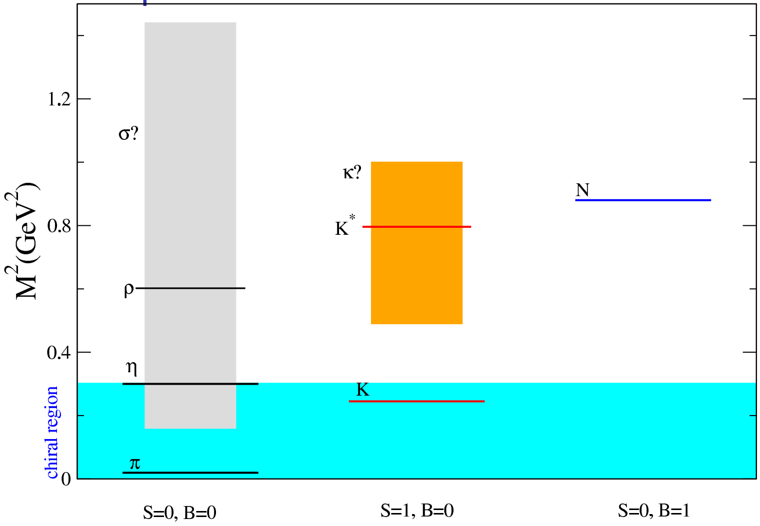
Zuoz 17. July 06

Outline

The QCD spectrum



The QCD spectrum

- the lowest-lying particles in the spectra are well understood: they would become exactly massless in the chiral limit of QCD (pseudo Goldstone bosons)
- the dynamics of strong interactions at low energy can be understood in terms of chiral symmetry
- the positions of the low-lying resonances is more difficult to determine and to understand
- they set the limit of validity of the chiral expansion – on the other hand they can be pinned down quite precisely thanks to the chiral expansion! cf. Leutwyler's talk

Systems with spontaneous symmetry breaking

- If a symmetry is spontaneously broken the spectrum contains massless particles – the Goldstone bosons
- Symmetry constrains the interactions of the Goldstone bosons – their interactions vanishes at low energy
- Green functions contain poles and cuts due to the exchange of Goldstone bosons – the vertices, on the other hand, can be expanded in powers of momenta. The coefficients of this expansion obey symmetry relations
- The effective Lagrangian is a systematic method to construct this expansion and in a way that respects these symmetry relations and all the general principles of quantum field theory
- The method leads to predictions – in some cases to very sharp ones

Quantum Chromodynamics in the chiral limit

$$\mathcal{L}^{(0)}_{QCD} = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \qquad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Large global symmetry group:

$$SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$$

1. $U(1)_V \Rightarrow$ baryonic number
2. $U(1)_A$ is anomalous
3. $$SU(3)_L \times SU(3)_R \Rightarrow SU(3)_V$$

   $\Rightarrow$ Goldstone bosons with the quantum numbers of pseudoscalar mesons will be generated

Quark masses, chiral expansion

In the real world quarks are not massless:

$$\mathcal{L}_{QCD} = \mathcal{L}^{(0)}_{QCD} + \mathcal{L}_m, \qquad \mathcal{L}_m := -\bar{q}\mathcal{M}q$$

$$\mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

the mass term $\mathcal{L}_m$ can be considered as a small perturbation $\Rightarrow$
Expand around $\mathcal{L}^{(0)}_{QCD}$ $\equiv$ Expand in powers of m_q

Chiral perturbation theory, the low-energy effective theory of QCD, is a simultaneous expansion in powers of momenta and quark masses

Quark mass expansion of meson masses

General quark mass expansion for the P particle:

$$M_P^2 = M_0^2 + \langle P|\bar{q}\mathcal{M}q|P\rangle + O(m_q^2)$$

For the pion $M_0^2 = 0$:

$$M_\pi^2 = -(m_u + m_d)\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle + O(m_q^2)$$

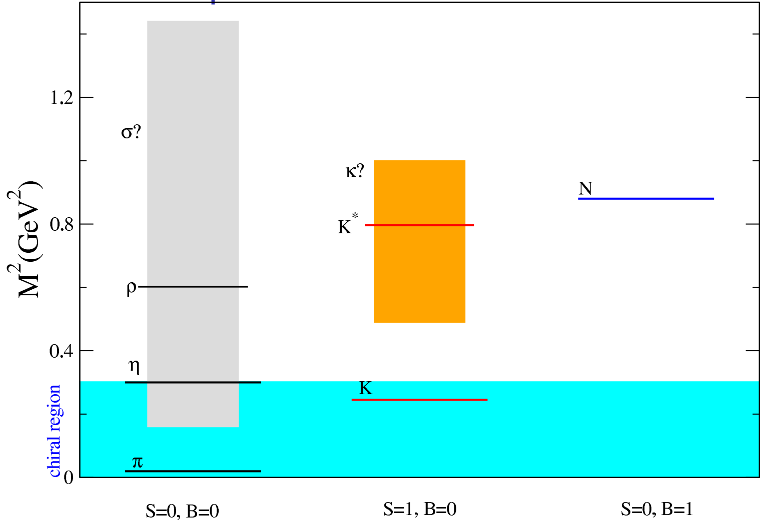
where we have used a Ward identity:

$$\langle \pi|\bar{q}q|\pi\rangle = -\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle =: B_0$$

$\langle 0|\bar{q}q|0\rangle$ is an order parameter for the chiral spontaneous symmetry breaking

Gell-Mann, Oakes and Renner (68)

Quark mass expansion of meson masses



Goldstone theorem

Be $\mathcal{H}$ a Hamiltonian symmetric under the group of transformations G: [Q_i are the generators of G]

$$[Q_i, \mathcal{H}] = 0 \qquad\qquad i = 1, \dots n_G$$

Be the ground state not invariant under G, i.e. for some generators X_i

$$X_i|0\rangle \neq 0$$

$$\{Q_1, \dots, Q_{n_G}\} = \{H_1, \dots, H_{n_H}, X_1, \dots, X_{n_G - n_H}\}$$

Goldstone theorem

$$[Q_i, \mathcal{H}] = 0 \qquad i = 1, \ldots n_G \ , \qquad X_i|0\rangle \neq 0 \ , \qquad H_i|0\rangle = 0$$

1. The subset of generators H_i which annihilate the vacuum forms a subalgebra

$$[H_i, H_k]|0\rangle = 0 \qquad\qquad i, k = 1, \ldots n_H$$

2. The spectrum of the theory contains $n_G - n_H$ massless excitations

$$X_i|0\rangle \qquad\qquad i = 1, \ldots n_G - n_H$$

from $[X_i, \mathcal{H}] = 0$ follows that $X_i|0\rangle$ is an eigenstate of the Hamiltonian with the same eigenvalue as the vacuum

Goldstone theorem

$$[Q_i, \mathcal{H}] = 0 \qquad i = 1, \dots n_G \ , \qquad X_i|0\rangle \neq 0 \ , \qquad H_i|0\rangle = 0$$

- $X_i|0\rangle$ are the Goldstone boson states
- the X_i are generators of the quotient space G/H
- the Goldstone fields are elements of the space G/H
- their transformation properties under G are fully dictated
- the dynamics of the Goldstone bosons at low energy is strongly constrained by symmetry

Matrix elements of conserved currents

Goldstone's theorem also asserts the following:

Take the transition matrix elements between the conserved currents associated with the generators Q_i and the Goldstone bosons

$$\langle 0|J_i^\mu|\pi^a(p)\rangle = iF_i^a p^\mu$$

The $n_G \times (n_G - n_H)$ matrix F_i^a has rank $N_{GB} = n_G - n_H$

We have introduced the symbol π for the Goldstone boson fields, and will call them "pions", as in strong interactions. Our arguments, however, will remain completely general

Pions do not interact at low energy

Current conservation implies

$$p_\mu \langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\ldots \mathrm{out}|J_i^\mu|0\rangle = 0 \qquad p^\mu = p_1^\mu + p_2^\mu + \ldots$$

Consider the amplitude for pair creation

$$\langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\mathrm{out}|J_i^\mu|0\rangle = \frac{p_3^\mu}{p_3^2}\sum_{a_3} F_i^{a_3} v_{a_1 a_2 a_3}(p_i) + \ldots$$

$$\text{Current conserv.} \Rightarrow \sum_{a_3} F_i^{a_3} v_{a_1 a_2 a_3}(0) = 0 \Rightarrow v_{a_1 a_2 a_3}(0) = 0$$

Because of Lorentz invariance, the function $v_{a_1 a_2 a_3}(p_1, p_2, p_3)$ can only depend on p_1^2, p_2^2, p_3^2: on the mass shell it is always zero

Low energy expansion

- Symmetry implies that Goldstone bosons do not interact at low energy

Low energy expansion

- Symmetry implies that Goldstone bosons do not interact at low energy
- If we take explicitly into account the poles in the Green functions which are due to exchanges of Goldstone bosons we can expand the vertices in powers of momenta
- The symmetry of the system implies also relations among the coefficients in the Taylor expansion in the momenta
- The effective Lagrangian is a systematic method to construct this expansion in a way that automatically respects the symmetry of the system
- Effective Lagrangian for Goldstone Bosons = CHPT

Transformation properties of the pions

The pion fields transform according to a representation of G

$$g \in G : \vec{\pi} \to \vec{\pi}' = \vec{f}(g, \vec{\pi})$$

where f has to obey the composition law

$$\vec{f}(g_1, \vec{f}(g_2, \vec{\pi})) = \vec{f}(g_1 g_2, \vec{\pi})$$

Consider the image of the origin $\vec{f}(g, 0)$: the elements which leave the origin invariant form a subgroup – the conserved subgroup H

$\vec{f}(gh, 0)$ coincides with $\vec{f}(g, 0)$ for each $g \in G$ and $h \in H \Rightarrow$ the function $\vec{f}$ maps elements of G/H onto the space of pion fields

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$$\vec{f}(g_1, \vec{f}(g_2, \vec{\pi})) = \vec{f}(g_1 g_2, \vec{\pi})$$

The mapping is invertible: $\vec{f}(g_1, 0) = \vec{f}(g_2, 0)$ implies $g_1 g_2^{-1} \in H$
$\Rightarrow$ pions can be identified with elements of G/H

Action of G on G/H

Two elements of G, $g_{1,2}$ are identified with the same element of G/H if

$$g_1 g_2^{-1} \in H$$

Let us call q_i the elements of G/H

The action of G on G/H is given by

$$g q_1 = q_2 h \qquad \text{where} \quad h(g, q_1) \in H$$

The transformation properties of the coordinates of G/H under the action of G are nonlinear (h is in general a nonlinear function of q_1 and g)

The space G/H for QCD

The choice of a representative element inside each equivalence class is arbitrary. For example

$$g = (g_L, g_R) = (1, g_R g_L^{-1}) \cdot (g_L, g_L) =: q \cdot h$$

but also $$g = (g_L, g_R) = (g_L g_R^{-1}, 1) \cdot (g_R, g_R) =: q' \cdot h'$$

where $$q, q' \in G/H \quad \text{and} \quad h, h' \in H$$

Action of G on G/H

$$\begin{aligned}(V_L, V_R) \cdot (1, g_R g_L^{-1}) &= (V_L, V_R g_R g_L^{-1}) \\ &= (1, V_R g_R g_L^{-1} V_L^{-1}) \cdot (V_L, V_L)\end{aligned}$$

The space G/H for QCD

In the literature the pion fields are usually collected in a matrix–valued field U, which transforms like

$$U \xrightarrow{G} U' = V_R U V_L^{-1}$$

U is nothing but a shorthand notation for $(1, g_R g_L^{-1})$, or its non-trivial part $g_R g_L^{-1}$

As a matrix U is a member of $SU(3)$, and therefore it can be written as

$$U = e^{i\phi^a \lambda_a}$$

where ϕ^a are the eight pion fields

Construction of the effective Lagrangian

In order to reproduce the low–energy structure of QCD we construct an effective Lagrangian which:

- contains the pion fields as the only degrees of freedom
- is invariant under G
- and expand it in powers of momenta

$$\begin{aligned}\mathcal{L}_{eff} &= f_1(U) + f_2(U)\langle U^+ \Box U\rangle \\ &+ f_3(U)\langle \partial_\mu U^+ \partial^\mu U\rangle + O(p^4)\end{aligned}$$

The invariance under transformations $U \xrightarrow{G} U' = V_R U V_L^{-1}$ implies that $f_{1,2,3}(U)$ do not depend on $U \Rightarrow f_1$ can simply be dropped, as it is an irrelevant constant

Construction of the effective Lagrangian

Using partial integration we end up with

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots \qquad \mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U^+ \partial^\mu U \rangle$$

where we have fixed the constant in front of the trace by looking at the Noether currents of the G symmetry:

$$V_i^\mu = i\frac{F^2}{4}\langle \lambda_i [\partial^\mu U, U^+] \rangle \qquad A_i^\mu = i\frac{F^2}{4}\langle \lambda_i \{\partial^\mu U, U^+\} \rangle$$

and comparing the result of the matrix element with the definition

$$\langle 0 | A_i^\mu | \pi^k(p) \rangle = i p^\mu \delta_{ik} F$$

Some more details

The matrix field U is an exponential of the pion fields π. If we want fields π of canonical dimension, we have to introduce a dimensional constant in the definition of U:

$$U = \exp\left\{ \frac{i}{F'} \pi^k \lambda_k \right\}$$

The requirement that the kinetic term of the pion fields is standard:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} \partial_\mu \pi^i \partial^\mu \pi^i \qquad \text{implies:} \qquad F = F'$$

The Lagrangian contains only one coupling constant which is the pion decay constant

The first prediction: $\pi\pi$ scattering

Isospin invariant amplitude:

$$M(\pi^a\pi^b \to \pi^c\pi^d) = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,u,s) + \delta_{ad}\delta_{bc}A(u,s,t)$$

Using the effective Lagrangian above

$$A(s,t,u) = \frac{s}{F^2}$$

Exercise: calculate it!

CHPT and explicit symmetry breaking?

- The effective Lagrangian was constructed in order to systematically account for symmetry relations.
  If the symmetry is explicitly broken can we still use it?
- If the symmetry breaking is weak we can make a perturbative expansion: matrix elements of the symmetry breaking Lagrangian (or of powers thereof) will appear
- Once we know the transformation properties of the symmetry breaking term, we can use symmetry to constrain its matrix elements
- The effective Lagrangian is still the appropriate tool to be used if we want to derive systematically all symmetry relations

Effective Lagrangian with ESB

$$\mathcal{L}^{\mathrm{QCD}} = \mathcal{L}_0^{\mathrm{QCD}} - \bar{q}\mathcal{M}q$$

The symmetry breaking term

$$\bar{q}\mathcal{M}q = \bar{q}_R\mathcal{M}q_L + \text{h.c.}$$

becomes also chiral invariant if we impose that the quark mass matrix $\mathcal{M}$ transforms according to

$$\mathcal{M} \to \mathcal{M}' = V_R \mathcal{M} V_L^+$$

We can now proceed to construct a chiral invariant effective Lagrangian that includes explicitly the matrix $\mathcal{M}$:

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M})$$

Leading order effective Lagrangian

The complete leading order effective Lagrangian of QCD reads:

$$\mathcal{L}_2 = \frac{F^2}{4}\left[\langle \partial_\mu U^+ \partial^\mu U \rangle + \langle 2B\mathcal{M}\left(U + U^+\right)\rangle\right]$$

F is the pion decay constant in the chiral limit

B is related to the $\bar{q}q$–condensate and to the pion mass

$$M_\pi^2 = 2B\hat{m} + O(\hat{m}^2)$$

$\pi\pi$ scattering to leading order

In the presence of quark masses the $\pi\pi$ scattering amplitude becomes

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}$$

Weinberg (66)

The two S–wave scattering lengths read

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16 \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045$$

The chiral Lagrangian to higher orders

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots$$

$\mathcal{L}_2$	contains	$(2, 2)$	constants	
$\mathcal{L}_4$	contains	$(7, 10)$	constants	Gasser, Leutwyler (84)
$\mathcal{L}_6$	contains	$(53, 90)$	constants	Bijnens, GC, Ecker (99)

The number in parentheses are for an $SU(N)$ theory with $N = (2, 3)$

The $\mathcal{L}_4$ Lagrangian

$$
\begin{aligned}
\mathcal{L}_4 \;=\;& L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
&+L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&-iL_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \\
&+L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}
$$

$$
\begin{aligned}
D_\mu U &= \partial_\mu U - i r_\mu U + i U l_\mu \qquad \chi = 2B(s+ip) \\
F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \\
r_\mu &= v_\mu + a_\mu \qquad l_\mu = v_\mu - a_\mu
\end{aligned}
$$

Summary

- I have discussed Goldstone's theorem and some of its physical implications at low energy
- The effective Lagrangian for Goldstone bosons is a tool to derive systematically the consequences of the symmetry on their interactions – I have discussed the principles that allow one to construct it
- The effective Lagrangian is useful also in the presence of a (small) explicit symmetry breaking – I have shown how to construct it even in this case