# Introduction to chiral perturbation theory
# I Foundations

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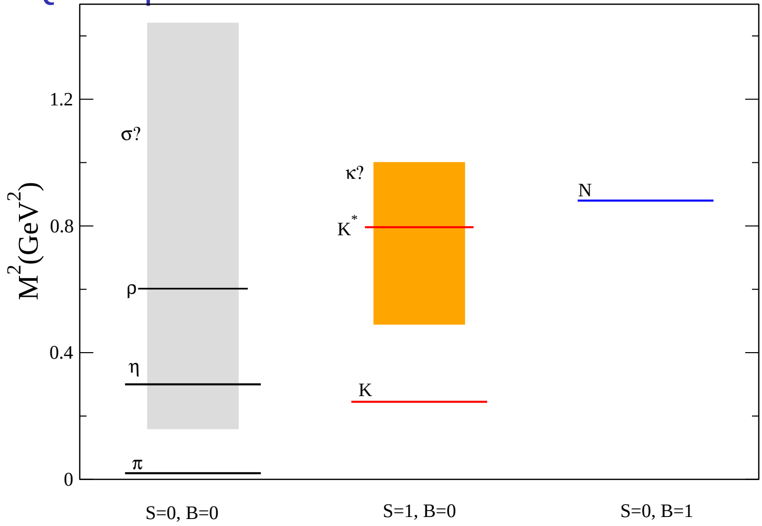
# Outline

# The QCD spectrum



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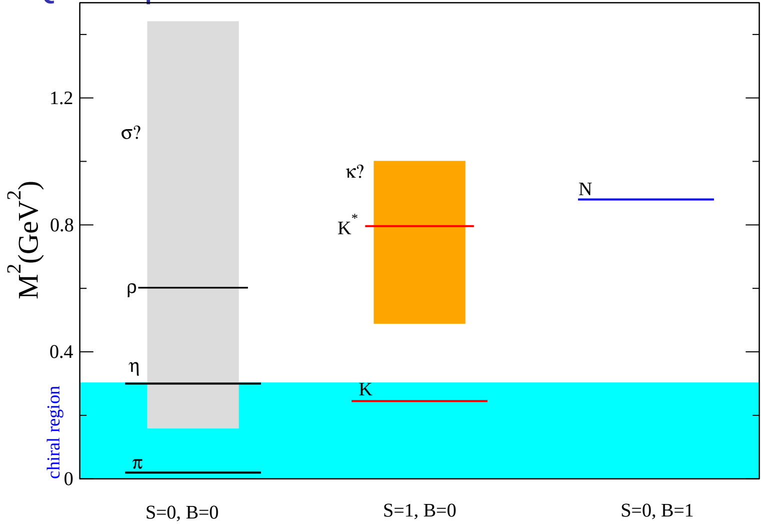

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- the dynamics of strong interactions at low energy can be understood in terms of chiral symmetry
- the positions of the low-lying resonances is more difficult to determine and to understand
- they set the limit of validity of the chiral expansion – on the other hand they can be pinned down quite precisely thanks to the chiral expansion! cf. Leutwyler's talk

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- The effective Lagrangian is a systematic method to construct this expansion and in a way that respects these symmetry relations and all the general principles of quantum field theory
- The method leads to predictions
  – in some cases to very sharp ones

# Quantum Chromodynamics in the chiral limit

$$\mathcal{L}_{QCD}^{(0)} = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \qquad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Large global symmetry group:

$$SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$$

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1. $U(1)_V \Rightarrow$ baryonic number
2. $U(1)_A$ is anomalous
3. $$SU(3)_L \times SU(3)_R \Rightarrow SU(3)_V$$

   $\Rightarrow$ Goldstone bosons with the quantum numbers of pseudoscalar mesons will be generated

# Quark masses, chiral expansion

In the real world quarks are not massless:

$$\mathcal{L}_{QCD} = \mathcal{L}^{(0)}_{QCD} + \mathcal{L}_m, \qquad \mathcal{L}_m := -\bar{q}\mathcal{M}q$$

$$\mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

the mass term $\mathcal{L}_m$ can be considered as a small perturbation $\Rightarrow$

Expand around $\mathcal{L}^{(0)}_{QCD}$ $\equiv$ Expand in powers of $m_q$

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Chiral perturbation theory, the low-energy effective theory of QCD, is a simultaneous expansion in powers of momenta and quark masses

# Quark mass expansion of meson masses

General quark mass expansion for the $P$ particle:

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where we have used a Ward identity:

$$\langle \pi|\bar{q}q|\pi\rangle = -\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle =: B_0$$

$\langle 0|\bar{q}q|0\rangle$ is an order parameter for the chiral spontaneous symmetry breaking

Gell-Mann, Oakes and Renner (68)

# Quark mass expansion of meson masses

Consider the whole pseudoscalar octet:

$$\begin{aligned}
M_\pi^2 &= (m_u + m_d)B_0 + O(m_q^2) \\
M_{K^+}^2 &= (m_u + m_s)B_0 + O(m_q^2) \\
M_{K^0}^2 &= (m_d + m_s)B_0 + O(m_q^2) \\
M_\eta^2 &= \frac{1}{3}(m_u + m_d + 4m_s)B_0 + O(m_q^2)
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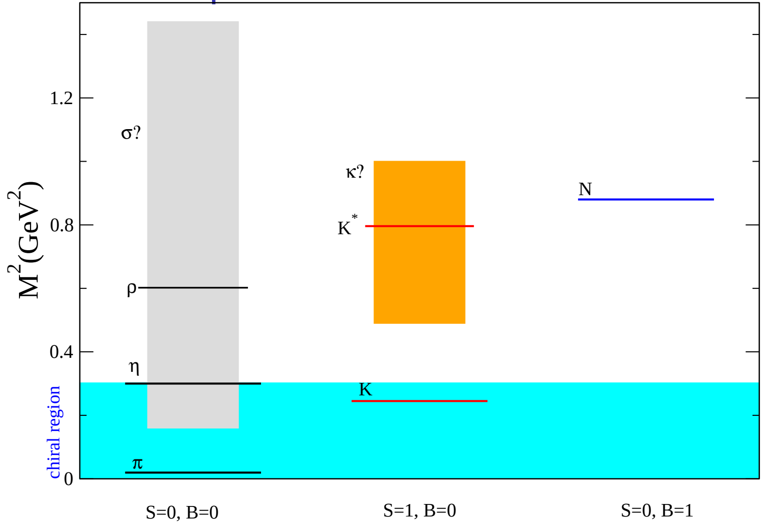
Consequences: $(\hat{m} = (m_u + m_d)/2)$

$$\begin{aligned}
M_K^2/M_\pi^2 &= (m_s + \hat{m})/2\hat{m} && \Rightarrow m_s/\hat{m} = 25.9 \\
M_\eta^2/M_\pi^2 &= (2m_s + \hat{m})/3\hat{m} && \Rightarrow m_s/\hat{m} = 24.3 \\
3M_\eta^2 &= 4M_K^2 - M_\pi^2 && \text{Gell-Mann–Okubo (62)} \\
(0.899 &= 0.960)\ \text{GeV}^2
\end{aligned}$$

# Quark mass expansion of meson masses



# Goldstone theorem

Be $\mathcal{H}$ a Hamiltonian symmetric under the group of transformations $G$: [$Q_i$ are the generators of $G$]

$$[Q_i, \mathcal{H}] = 0 \qquad\qquad i = 1, \ldots n_G$$

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Be the ground state not invariant under $G$, i.e. for some generators $X_i$

$$X_i|0\rangle \neq 0$$

$$\{Q_1, \ldots, Q_{n_G}\} = \{H_1, \ldots, H_{n_H}, X_1, \ldots, X_{n_G - n_H}\}$$

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1. The subset of generators $H_i$ which annihilate the vacuum forms a subalgebra

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1. The subset of generators $H_i$ which annihilate the vacuum forms a subalgebra

$$[H_i, H_k]|0\rangle = 0 \qquad\qquad i, k = 1, \ldots n_H$$

2. The spectrum of the theory contains $n_G - n_H$ massless excitations

$$X_i|0\rangle \qquad\qquad i = 1, \ldots n_G - n_H$$

from $[X_i, \mathcal{H}] = 0$ follows that $X_i|0\rangle$ is an eigenstate of the Hamiltonian with the same eigenvalue as the vacuum

# Goldstone theorem

$$[Q_i, \mathcal{H}] = 0 \qquad i = 1, \ldots n_G \;, \qquad X_i|0\rangle \neq 0 \;, \qquad H_i|0\rangle = 0$$

- $X_i|0\rangle$ are the Goldstone boson states
- the $X_i$ are generators of the quotient space $G/H$
- the Goldstone fields are elements of the space $G/H$
- their transformation properties under $G$ are fully dictated
- the dynamics of the Goldstone bosons at low energy is strongly constrained by symmetry

# Matrix elements of conserved currents

Goldstone's theorem also asserts the following:

Take the transition matrix elements between the conserved currents associated with the generators $Q_i$ and the Goldstone bosons

$$\langle 0|J_i^\mu|\pi^a(p)\rangle = iF_i^a p^\mu$$

The $n_G \times (n_G - n_H)$ matrix $F_i^a$ has rank $N_{GB} = n_G - n_H$

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We have introduced the symbol $\pi$ for the Goldstone boson fields, and will call them "pions", as in strong interactions. Our arguments, however, will remain completely general

# Pions do not interact at low energy

Current conservation implies

$$p_\mu \langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\dots\text{out}|J_i^\mu|0\rangle = 0 \qquad p^\mu = p_1^\mu + p_2^\mu + \dots$$

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Consider the amplitude for pair creation

$$\langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\mathrm{out}|J_i^\mu|0\rangle = \frac{p_3^\mu}{p_3^2}\sum_{a_3} F_i^{a_3} v_{a_1 a_2 a_3}(p_i) + \ldots$$

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Current conserv. $\Rightarrow \sum_{a_3} F_i^{a_3} v_{a_1a_2a_3}(0) = 0 \Rightarrow v_{a_1a_2a_3}(0) = 0$

Because of Lorentz invariance, the function $v_{a_1a_2a_3}(p_1, p_2, p_3)$ can only depend on $p_1^2$, $p_2^2$, $p_3^2$: on the mass shell it is always zero

# Pions do not interact at low energy

Now consider the amplitude for three–pion creation from a conserved current

$$\langle \pi^{a_1}\pi^{a_2}\pi^{a_3}\text{out}|J_i^{\mu}|0\rangle = \frac{p_4^{\mu}}{p_4^2}\sum_{a_4} F_i^{a_4} v_{a_1a_2a_3a_4}(p_i) + \ldots$$

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In this case the vertex function can depend on two Lorentz scalars, $s$, and $t$, and we can do a Taylor expansion:

$$v_{a_1a_2a_3a_4}(p_1,p_2,p_3,p_4) = c^1_{a_1a_2a_3a_4} s + c^2_{a_1a_2a_3a_4} t + \dots$$

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- Effective Lagrangian for Goldstone Bosons = CHPT

# Transformation properties of the pions

The pion fields transform according to a representation of $G$

$$g \in G : \vec{\pi} \to \vec{\pi}' = \vec{f}(g, \vec{\pi})$$

where $f$ has to obey the composition law

$$\vec{f}(g_1, \vec{f}(g_2, \vec{\pi})) = \vec{f}(g_1 g_2, \vec{\pi})$$

Consider the image of the origin $\vec{f}(g, 0)$: the elements which leave the origin invariant form a subgroup – the conserved subgroup $H$

$\vec{f}(gh, 0)$ coincides with $\vec{f}(g, 0)$ for each $g \in G$ and $h \in H \Rightarrow$ the function $\vec{f}$ maps elements of $G/H$ onto the space of pion fields

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The mapping is invertible: $\vec{f}(g_1, 0) = \vec{f}(g_2, 0)$ implies $g_1 g_2^{-1} \in H$
$\Rightarrow$ pions can be identified with elements of $G/H$

# Action of $G$ on $G/H$

Two elements of $G$, $g_{1,2}$ are identified with the same element of $G/H$ if

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The transformation properties of the coordinates of $G/H$ under the action of $G$ are nonlinear ($h$ is in general a nonlinear function of $q_1$ and $g$)

# The space $G/H$ for QCD

The choice of a representative element inside each equivalence class is arbitrary. For example

$$g = (g_L, g_R) = (1, g_R g_L^{-1}) \cdot (g_L, g_L) =: q \cdot h$$

but also $$g = (g_L, g_R) = (g_L g_R^{-1}, 1) \cdot (g_R, g_R) =: q' \cdot h'$$

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**Action of $G$ on $G/H$**

$$\begin{aligned}(V_L, V_R) \cdot (1, g_R g_L^{-1}) &= (V_L, V_R g_R g_L^{-1}) \\ &= (1, V_R g_R g_L^{-1} V_L^{-1}) \cdot (V_L, V_L)\end{aligned}$$

# The space $G/H$ for QCD

In the literature the pion fields are usually collected in a matrix–valued field $U$, which transforms like

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As a matrix $U$ is a member of $SU(3)$, and therefore it can be written as

$$U = e^{i\phi^a \lambda_a}$$

where $\phi^a$ are the eight pion fields

# Construction of the effective Lagrangian

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- and expand it in powers of momenta

$$\begin{aligned}\mathcal{L}_{eff} &= f_1(U) + f_2(U)\langle U^+ \Box U\rangle \\ &+ f_3(U)\langle \partial_\mu U^+ \partial^\mu U\rangle + O(p^4)\end{aligned}$$

The invariance under transformations $U \xrightarrow{G} U' = V_R U V_L^{-1}$ implies that $f_{1,2,3}(U)$ do not depend on $U \Rightarrow f_1$ can simply be dropped, as it is an irrelevant constant

# Construction of the effective Lagrangian

Using partial integration we end up with

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \qquad \mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U^+ \partial^\mu U \rangle$$

where we have fixed the constant in front of the trace by looking at the Noether currents of the $G$ symmetry:

$$V_i^\mu = i\frac{F^2}{4}\langle \lambda_i [\partial^\mu U, U^+] \rangle \qquad A_i^\mu = i\frac{F^2}{4}\langle \lambda_i \{\partial^\mu U, U^+\} \rangle$$

and comparing the result of the matrix element with the definition

$$\langle 0 | A_i^\mu | \pi^k(p) \rangle = i p^\mu \delta_{ik} F$$

# Some more details

The matrix field $U$ is an exponential of the pion fields $\pi$. If we want fields $\pi$ of canonical dimension, we have to introduce a dimensional constant in the definition of $U$:

$$U = \exp\left\{\frac{i}{F'}\pi^k\lambda_k\right\}$$

The requirement that the kinetic term of the pion fields is standard:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\partial_\mu\pi^i\partial^\mu\pi^i \qquad \text{implies:} \qquad F = F'$$

The Lagrangian contains only one coupling constant which is the pion decay constant

# The first prediction: $\pi\pi$ scattering

Isospin invariant amplitude:

$$M(\pi^a\pi^b \to \pi^c\pi^d) = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,u,s) + \delta_{ad}\delta_{bc}A(u,s,t)$$

Using the effective Lagrangian above

$$A(s,t,u) = \frac{s}{F^2}$$

Exercise: calculate it!

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- If the symmetry breaking is weak we can make a perturbative expansion: matrix elements of the symmetry breaking Lagrangian (or of powers thereof) will appear
- Once we know the transformation properties of the symmetry breaking term, we can use symmetry to constrain its matrix elements
- The effective Lagrangian is still the appropriate tool to be used if we want to derive systematically all symmetry relations

# Effective Lagrangian with ESB

$$\mathcal{L}^{\mathrm{QCD}} = \mathcal{L}_0^{\mathrm{QCD}} - \bar{q}\mathcal{M}q$$

The symmetry breaking term

$$\bar{q}\mathcal{M}q = \bar{q}_R\mathcal{M}q_L + \text{h.c.}$$

becomes also chiral invariant if we impose that the quark mass matrix $\mathcal{M}$ transforms according to

$$\mathcal{M} \to \mathcal{M}' = V_R\mathcal{M}V_L^+$$

# Effective Lagrangian with ESB

$$\mathcal{L}^{\text{QCD}} = \mathcal{L}_0^{\text{QCD}} - \bar{q}\mathcal{M}q$$

The symmetry breaking term

$$\bar{q}\mathcal{M}q = \bar{q}_R\mathcal{M}q_L + \text{h.c.}$$

becomes also chiral invariant if we impose that the quark mass matrix $\mathcal{M}$ transforms according to

$$\mathcal{M} \to \mathcal{M}' = V_R \mathcal{M} V_L^+$$

We can now proceed to construct a chiral invariant effective Lagrangian that includes explicitly the matrix $\mathcal{M}$:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M})$$

# Effective Lagrangian with ESB

To first order in $\mathcal{M}$ there is only one chiral invariant term which one can construct:

$$\mathcal{L}_{\mathcal{M}}^{(1)} = \frac{F^2}{2}\left[B\langle \mathcal{M}U^+\rangle + B^*\langle \mathcal{M}^+U\rangle\right]$$

Strong interactions respect parity $\Rightarrow$ $B$ must be real:

$$\mathcal{L}_{\mathcal{M}}^{(1)} = \frac{F^2 B}{2}\langle \mathcal{M}\left(U + U^+\right)\rangle$$

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Before using this Lagrangian: pin down the constant $B$:

$$B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle \qquad\qquad M_\pi^2 = 2B\hat{m}$$

# Leading order effective Lagrangian

The complete leading order effective Lagrangian of QCD reads:

$$\mathcal{L}_2 = \frac{F^2}{4}\left[\langle \partial_\mu U^+ \partial^\mu U \rangle + \langle 2B\mathcal{M}\left(U + U^+\right)\rangle\right]$$

$F$ is the pion decay constant in the chiral limit

$B$ is related to the $\bar{q}q$–condensate and to the pion mass

$$M_\pi^2 = 2B\hat{m} + O(\hat{m}^2)$$

# $\pi\pi$ scattering to leading order

In the presence of quark masses the $\pi\pi$ scattering amplitude becomes

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2} \qquad \text{Weinberg (66)}$$

The two $S$–wave scattering lengths read

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16 \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045$$

# The chiral Lagrangian to higher orders

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots$$

| | | | | |
|---|---|---|---|---|
| $\mathcal{L}_2$ | contains | $(2, 2)$ | constants | |
| $\mathcal{L}_4$ | contains | $(7, 10)$ | constants | Gasser, Leutwyler (84) |
| $\mathcal{L}_6$ | contains | $(53, 90)$ | constants | Bijnens, GC, Ecker (99) |

The number in parentheses are for an $SU(N)$ theory with $N = (2, 3)$

# The $\mathcal{L}_4$ Lagrangian

$$
\begin{aligned}
\mathcal{L}_4 \;=\;& L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
&+L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&-iL_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \\
&+L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}
$$

$$
\begin{aligned}
D_\mu U &= \partial_\mu U - i r_\mu U + i U l_\mu \qquad \chi = 2B(s+ip) \\
F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \\
r_\mu &= v_\mu + a_\mu \qquad l_\mu = v_\mu - a_\mu
\end{aligned}
$$

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- The effective Lagrangian for Goldstone bosons is a tool to derive systematically the consequences of the symmetry on their interactions – I have discussed the principles that allow one to construct it
- The effective Lagrangian is useful also in the presence of a (small) explicit symmetry breaking – I have shown how to construct it even in this case