Effective Field Theory: Concept, HQET, and SCET

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Effective Field Theory

– Concept,
Heavy quark effective theory,
Soft-collinear effective theory

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Quantum Field Theory

$$S = \int d^4x \; \mathcal{L}(\phi_{i(x)}, \partial_\mu \phi_{i(x)})$$

- ϕ_i so that spectrum of known particles / states is reproduced
  (Note: fields $\not\equiv$ particles)

- $\mathcal{L} = \sum_i c_i \mathcal{O}_i(x)$ field products at same x (locality)

  Makes it easy to satisfy Lorentz invariance + relativistic causality

- $\mathcal{L}$ restricted by internal symmetries

Quantum : $[\phi_i(t,\vec{x}), \pi_j(t,\vec{y})]_{\mp} = i\,\delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{y})$

Naturalness principle (less fundamental than above)

- all $\mathcal{O}_i$ compatible with field content and assumed symmetries should appear in $\mathcal{L}$

- $c_i = \dfrac{\lambda_i}{M^{d_i-4}}$ with $d_i = [\mathcal{O}_i] =$ mass dimension of $\mathcal{O}_i$

  λ_i dimensionless and $\mathcal{O}(1)$

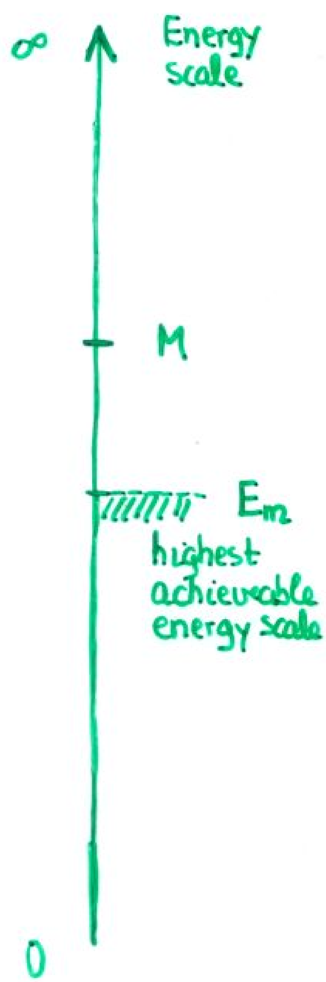
Fundamental vs. effective theories

Relativity + locality $\Rightarrow$ Sum over intermediate states contains sum over spectrum of arbitrarily high energies.

Technically

$$\text{Amplitude}(p_i) \sim \int_{\text{All } k} d^4k \, I(k, p_i)$$

Usually divergent $\Rightarrow$ renormalization



Fundamental :

- Adjust a finite number of couplings λ_i to experiment
- Then $\mathcal{L}$ provides an exact description up to arbitrarily high energies
  - far beyond E_m

Fundamental vs. effective theories

Relativity + locality $\Rightarrow$ Sum over intermediate states contains sum over spectrum of arbitrarily high energies. Technically

$$\text{Amplitude}(p_i) \sim \int_{\text{All } k} d^4k \, I(k, p_i)$$

Usually divergent $\Rightarrow$ renormalization

Effective :

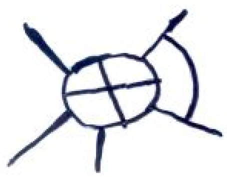
- Adjust a finite number of couplings to experiment (or: to a fundamental theory)

- Then $\mathcal{L}$ provides an approximate description as long as $E_m \ll M$.
  The approximation can be improved systematically by adding more couplings.

- For $E_m \gtrsim M$ must revise the description and turn to a new EFT (or: fundamental theory)

Super-, non-, and renormalizable interactions

Perturbative analysis



Rescale <u>all</u> loop momenta $k_i \to \lambda k_i$.
Take $\lambda \to \infty$

$$I \longrightarrow \lambda^D \left(I + \mathcal{O}(1/\lambda) \right)$$

superficial degree of divergence

e.g.

$$\int d^4k \quad \frac{1}{k^2} \, \frac{1}{(p+k)^2 - m^2}$$

$4 \quad -2 \quad -2$

$D = 0$ "logarithmically divergent"

$D > 0$ Diagram is certainly divergent

$$\mathcal{L} = \underset{(\partial_\mu \phi \partial^\mu \phi,\ \bar{\psi} i \gamma \psi, \dots)}{\text{kinetic terms}} + \sum_i \frac{\lambda_i}{M^{d_i - 4}} \mathcal{O}_i$$

L = # of loops
I_f = # of internal lines of field type f
E_f = external

V_i = # of Vertices of type i
a_i = # of derivatives in $\mathcal{O}_i$
n_{if} = # of fields of type f in $\mathcal{O}_i$

Propagator of f $\sim \dfrac{1}{(k^2)^{1-s_f}}$ for $k \gg m$

$$s_f = \begin{cases} 0 & \text{scalar field, massless vector field} \\ 1/2 & \text{spin-}1/2 \text{ field} \\ 1 & \text{massive vector field (general case)} \end{cases}$$

cause problems. Exclude this case.

$$D = \sum_f I_f (2s_f - 2) + \sum_i V_i a_i + 4L$$

$$L = \sum_f I_f - \sum_i V_i + 1$$ for connected diagrams

$$2I_f + E_f = \sum_i V_i n_{if}$$

$$d_i = a_i + \sum_f n_{if} \underbrace{(1+s_f)}_{}$$

= Mass dimension of field f (not valid for massive vector fields)

$$\Rightarrow \quad D = 4 - \sum_f E_f (1+s_f) + \sum_i V_i (d_i - 4)$$

Superrenormalizable interaction : $d_i < 4$

$D \downarrow$ with increasing V_i – only a finite number of diagrams is divergent

Renormalizable interaction : $d_i = 4$

D independent of V_i

$D < 0$ for diagrams with sufficiently many external lines

– only a finite number of counterterms / couplings is needed

$\Rightarrow$ Theories with $d_i \le 4 \ \forall i$ (= c_i dimensionless or positive mass dimension) are candidates for fundamental theories

Non-renormalizable interactions : $d_i > 4$

Diagrams with any number of external lines are divergent if vertex i occurs sufficiently often.

- must include all $\mathcal{O}_i$ as counterterms / couplings up to arbitrary dimension

$$\mathcal{L} = \text{kinetic terms} + \sum_i \frac{\lambda_i}{M^{d_i - 4}} \mathcal{O}_i$$

Consider scattering amplitude with mass dimension A and a diagram with n insertions of $\mathcal{O}_i$ with $d_i > 4$. Then the contribution is (use dimensional regularization) of order

$$\lambda_i^n \cdot E_m^A \left(\frac{E_m}{M}\right)^{n \cdot (d_i - 4)}$$

E_m = scale of external momenta

$\Rightarrow$ only a finite number of non-renormalizable interactions is really relevant for $E_m \ll M$

Effective theories are non-renormalizable theories which can be used as long as $E_m \ll M \approx$ scale of non-renormalizable interactions.

[Note: non-renormalizable theories according to this analysis may be fundamental if they have "UV-fixed points". The problem is non-perturbative.]

Remarks

- The effect of intermediate states with $E \gtrsim M$ is local in experiments at energies $\lesssim E_m$

  Change in "Ultraviolet Physics" $\Rightarrow$ change in couplings λ_i in $\sum_i \frac{\lambda_i}{M^{d_i-4}} \mathcal{O}_i$

  Since $\mathcal{L}$ includes all possible $\mathcal{O}_i$ and since λ_i is determined from data, the description is UV - insensitive

  In this sense, the high energy fluctuations are "integrated out" and reside in the values of the λ_i

- Can decide theoretically whether a theory can be fundamental (see above).
  But can never decide by experiment whether nature is really described by this fundamental theory at all energies.

  False fundamental theories cannot always be falsified (since $E \to \infty$)

Historical note

Old-fashioned renormalization paradigm ($\lesssim$ 1980-1990)

Good theories are renormalizable.

Non-renormalizable theories are BAD (unpredictive ...)

Modern renormalization / EFT paradigm ($\gtrsim$ 1980-1990)

Most theories are probably effective theories and non-renormalizable.

Super-renormalizable interactions are BAD and should be forbidden by symmetries.

Example: mass term for a scalar field

$$\mathcal{L} \supset c_2 \phi^2 \quad \rightsquigarrow \quad c_2 = \lambda_2 M^2$$

large scale $E_m \ll M$

$\longrightarrow$ Scalar mass is $\mathcal{O}(M)$, but then it is <u>not</u> included as dynamical field ↯

OR

$\longrightarrow$ $\lambda_2 \ll 1$ unnatural "fine-tuning"

Fundamental theory not known

Fundamental theory known

"bottom-up"

"top-down"

Effective Theory

The Standard Model

Einstein gravity

Higher-dimensional gauge theories

$\vdots$

Why is this useful?

- fundamental theory too difficult (QCD)
- emergent symmetries at $E \ll M$
- sum

$$\left(\lambda \log E/M\right)^n$$

can be large even for $\lambda \ll 1$

Integrating out top-quarks in QCD

$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \sum_{f=1}^{5} \bar{\psi}_f (i\not{D} - m_f)\psi_f$$

$$+ \bar{Q}(i\not{D} - m_Q)Q \qquad Q \text{ top field}$$

Assumption: $p_i \cdot p_j \ll m_t^2$ for all external momenta

$\Rightarrow$ no external Q lines

$$A^\mu = A^{\mu(L)} + A^{\mu(H)}$$
$$\psi_f = \psi_f^{(L)} + \psi_f^{(H)}$$

low- and high frequencies

$$Z[J,\eta,\bar{\eta}] = N \int D[A,\psi_f,\bar{\psi}_f,Q,\bar{Q}]\, e^{i\int d^4x \left(\mathcal{L} + J^\mu A_\mu^{(L)} + \bar{\eta}\psi_f^{(L)} + \bar{\psi}_f^{(L)}\eta\right)}$$

$$= N' \int D[A^{(L)},\psi_f^{(L)},\bar{\psi}_f^{(L)}]\, e^{i\int d^4x \left(\mathcal{L}_{eff} + J^\mu A_\mu^{(L)} + \bar{\eta}\psi_f^{(L)} + \bar{\psi}_f^{(L)}\eta\right)}$$

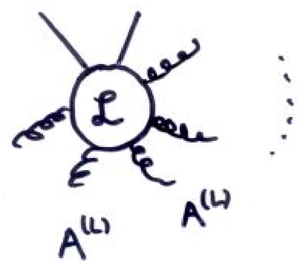
with

$$e^{iS_{eff}[A^{(L)},\psi_f^{(L)},\bar{\psi}_f^{(L)}]} = \frac{N}{N'} \int D[A^{(H)},\psi_f^{(H)},\bar{\psi}_f^{(H)},Q,\bar{Q}]\, e^{iS[A,\psi_f,\bar{\psi}_f,Q,\bar{Q}]}$$

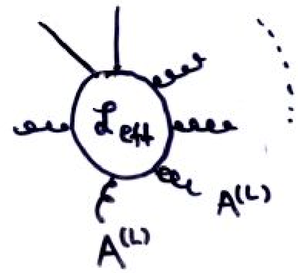
can be expanded in local operators

" integrating out the heavy fields and high energy modes "

In most cases S_{eff} can be constructed only perturbatively



$\stackrel{!}{=}$



Choose $\lambda_i O_i$
so that this
is true

Consider gluon 2pt function

EXACTLY reproduced for $\mathcal{L}_{eff} = -\frac{1}{4} G^2 + \sum_f \bar{\Psi}_f (i\not{D} - m_f)\Psi_f$

Note: renormalize $\mathcal{L}$ and $\mathcal{L}_{eff}$ in $\overline{MS}$ after using dimensional regularization

No explicit high-frequency cut-off

High frequency modes of A, Ψ_f appear only in diagrams with Q- lines that contain the scale m_t

+ counterterm

$\stackrel{!}{=}$

Expansion in q^2/m_t^2

$$= i(q^2 g_{\mu\nu} - q_\mu q_\nu)\,\delta^{AB}\,\Pi(q^2)$$

$$\Pi(q^2) = + \frac{2 T_f \alpha_s}{\pi} \int_0^1 dx\, x(1-x) \ln \frac{m_t^2 - x(1-x) q^2}{\mu^2}$$

$T_f = \frac{1}{2}$

$$= \underbrace{\frac{\alpha_s T_f}{3\pi} \ln \frac{m_t^2}{\mu^2}}_{O_1 = -\frac{1}{4} G^2} - \underbrace{\frac{\alpha_s T_f}{15\pi} \frac{q^2}{m_t^2}}_{O_2 = G D_\mu D^\mu G} + \mathcal{O}\left(\frac{q^4}{m_t^4}\right)$$

$$\mathcal{L}_{eff} = -\frac{1}{4}\left(1 - \frac{\alpha_s T_f}{3\pi} \ln \frac{m_t^2}{\mu^2}\right) G^A_{\mu\nu} G^{A\mu\nu} + \sum_f \bar\Psi_f (i\not{D} - m_f)\Psi_f + \frac{\alpha_s T_f}{60\pi m_t^2} G^A_{\mu\nu} D^2 G^{A\mu\nu} + \dots$$

$d_i = 4$ term has been modified

$d_i = 6$ non-renormalizable interaction has been generated

Rescale gluon field to recover canonical kinetic term

$$\hat{A} = \left(1 - \frac{\alpha_s T_f}{6\pi} \ln \frac{m_t^2}{\mu^2}\right) A \quad \Rightarrow \quad -\frac{1}{4} \hat{G}^2$$

but

$$g_s \bar\Psi_f \not{A} \Psi_f = \underbrace{\frac{g_s}{1 - \frac{\alpha_s T_f}{6\pi} \ln \frac{m_t^2}{\mu^2}}}_{\equiv\, \hat{g}_s} \bar\Psi_f \hat{\not{A}} \Psi_f$$

$\equiv \hat{g}_s$ strong coupling in the effective theory

- use $\mathcal{L}$ $\qquad \mu^2 \frac{d\alpha_s}{d\mu^2} = -\beta_0 \frac{\alpha_s^2}{4\pi} \qquad \beta_0 = 11 - \frac{4}{3} \cdot 6$

- near m_t relate $\qquad \hat{\alpha}_s = \dfrac{\alpha_s}{1 - \frac{\alpha_s T_f}{3\pi} \ln \frac{m_t^2}{\mu^2}}$ $\qquad (*)$

$$\mu^2 \frac{d\hat{\alpha}_s}{d\mu^2} = \mu^2 \frac{d\alpha_s}{d\mu^2} + \frac{\alpha_s^2 T_f}{3\pi} \cdot (-1) = -\beta_0^{(5)} \frac{\hat{\alpha}_s^2}{4\pi} + O(\hat{\alpha}_s^3)$$

$$\beta_0^{(5)} = 11 - \frac{4}{3} \cdot 5$$

- far below m_t MUST use $\mathcal{L}_{eff}$
Otherwise for $\mu \sim p \ll m_t$ get $\alpha_s \ln \frac{m_t^2}{p^2}$
and perturbation theory breaks down.
In $\mathcal{L}_{eff}$ these high-energy logs are absorbed in $\hat{\alpha}_s(p)$

solve renormalization group eq.
$\mu^2 \frac{d\hat{\alpha}}{d\mu^2} = -\beta(\hat{\alpha})$ with
initial condition $(*)$

And so on .....

 $\longrightarrow \quad \frac{c}{m_t^2} f_{ABC} G^A_{\mu\nu} G^B_{\mu\alpha} G^C_{\nu\alpha}$

.....

Below m_b two options

- no external bottom lines $\rightarrow$ repeat above procedure

- external bottom lines must be conserved below m_b

$\longrightarrow$ Non-relativistic QCD, heavy quark effective theory

Effective theories of QCD

$\boxed{\text{QCD}}$

perturbative matching

QCD_{eff}
$[\Psi_f, A^\mu]$

QCD_{eff}
weakly coupled

(P)NRQCD for top

(P)NRQCD
SCET

High-temperature
High-density

QCD_{eff}
strongly coupled

HQET

non-perturbative matching

QCD_{eff}
[new fields]

- exploitation of symmetries
- effective theory <u>weakly</u> coupled

Chiral Lagrangian
Nucleon effective theory

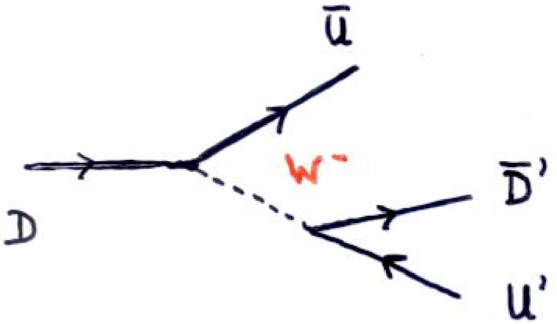
Effective weak interactions

[all details in Buras' lecture notes hep-ph/9806471]

Heavy quark decay due to weak interactions.

$m_b, m_c \ll M_W$

$$\Longrightarrow \quad \mathcal{L}_{SM} \longrightarrow \underbrace{\mathcal{L}_{QCD+QED}}_{\text{5-flavour}} + \underbrace{\mathcal{L}_{eff}}_{\substack{\text{effective weak}\\ \text{interactions}}}$$



$D, D' \in \{ d, s, b, e^-, \mu^-, \tau^- \}$

$U, U' \in \{ u, c, t, \nu_e, \nu_\mu, \nu_\tau \}$

- muon-decay
- neutron-decay

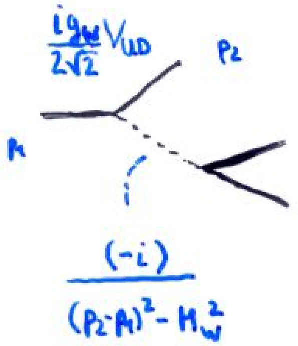
⋮

- $b \to c \ell \nu$ semi-leptonic heavy quark decay

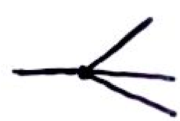
  $b \to c d \bar{u}$ hadronic "

⋮

Tree-level matching



$\longrightarrow$



$$- \frac{i g_W^2}{8 M_W^2} V_{UD} V^*_{U'D'} + \mathcal{O}\left(1/M_W^4\right)$$

$$\frac{G_F}{\sqrt{2}} \equiv \frac{g_W^2}{8 M_W^2}$$

- The leading term is non-renormalizable ($d_i = 6$). This explains why the weak interactions are weak at low energies. Effective coupling is

$$g_W^2 \cdot \left(\frac{E}{M_W}\right)^2$$

- $$\mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}} \sum V_{UD} V^*_{U'D'} \, \bar{U}\gamma^\mu(1-\gamma_5)D \, \bar{D}'\gamma_\mu(1-\gamma_5)U'$$

  Fermi's theory

- Higher-dimensional operators give $\left(\frac{m_b}{M_W}\right)^2 \ll 1\%$ corrections to b decay – usually neglected

Beyond tree level

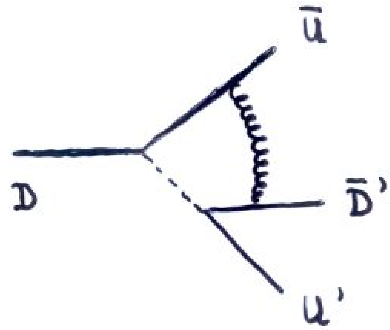
- Add all gauge-invariant operators up to dimension 6 with undetermined couplings $C_i(\mu)$

$$\mathcal{L}_{eff} = \frac{G_F}{\sqrt{2}} \sum_i C_i(\mu) O_i$$

- Determine C_i by matching suitable amplitudes

$$A|_{SM} \stackrel{!}{=} A|_{\substack{QCD\\QED} + \mathcal{L}_{eff}}$$

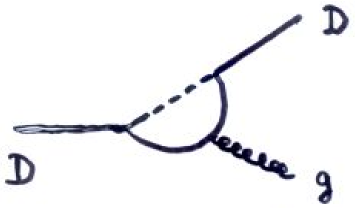
in perturbation theory

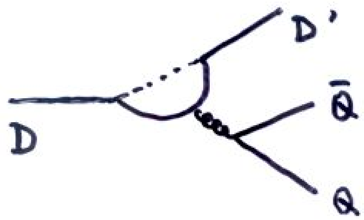


$$\mathcal{O}_1 = [\bar{U}_a D_a]_{V-A}\,[\bar{D}'_b U'_b]_{V-A}$$

$$\mathcal{O}_2 = [\bar{U}_a D_b]_{V-A}\,[\bar{D}'_b U'_a]_{V-A}$$

colour



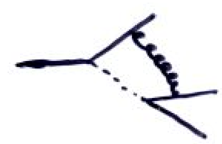


$$m_D\,\bar{D}'_a \sigma^{\mu\nu} G^A_{\mu\nu} T^A_{ab} (1+\gamma_5) D_b$$

$$[\bar{Q}Q]\,[D'D]$$

with various Dirac matrices + colour contractions

$$\text{tree} + \alpha_s \int d^4k\ \frac{1}{k^2 - M_W^2}\ I(p_i, k)$$

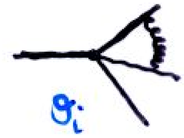


cannot simply expand in M_W ; non-analytic from $k \sim M_W$

$$\sim\ 1 + \alpha_s \ln \frac{M_W^2}{p^2}$$

$$= \left(1 + \alpha_s \ln \frac{M_W^2}{\mu^2}\right)\left(1 + \alpha_s \ln \frac{\mu^2}{p^2}\right) + \cdots$$

non-analytic from $k \sim M_W$



$C_i(\mu)$

$\mathcal{O}_i$

independent of low-energy scale p

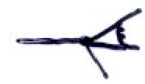
This loop is UV-divergent contrary to SM loop, but independent of M_W

$\hookrightarrow \log \frac{\mu^2}{p^2}$

Explicitly

$$C_1(\mu) = 1 + \frac{\alpha_s(\mu)}{4\pi}\left(\ln\frac{M_W^2}{\mu^2} - \frac{11}{6}\right) + O(\alpha_s^2)$$

$$C_2(\mu) = \frac{\alpha_s(\mu)}{4\pi}\left(-3\ln\frac{M_W^2}{\mu^2} + \frac{11}{2}\right) + O(\alpha_s^2) \qquad (*)$$

Want to evaluate at $\mu \sim p$ so that  contains no large log μ^2/p^2.

$$\mu\frac{d}{d\mu}C_i(\mu) = \underbrace{\gamma_{ji}(\alpha_s)}_{\text{anomalous dimension}}C_j(\mu) \qquad (**)$$

Solve this to determine $C_i(\mu_{low})$ given $C_i(\mu_{high})$ from $(*)$.
Sums all

$$\sum_n \alpha_s^{k-1}\left(\alpha_s \ln\frac{M_W^2}{\mu_{low}^2}\right)^n$$

if anomalous dimension is computed to k loops

Example:

$C_i(\mu_{high})$ tree-level $C_1 = 1$, $C_2 = 0$ + 1 loop anomalous dimension $\Longrightarrow$ $C_1(m_b) \approx 1.12$, $C_2(m_b) \approx -0.27$

$$\mathcal{L}_{\substack{QCD_5+\\QED}} + \mathcal{L}_{eff}^{weak}$$

can be derived once and forever.

THIS effective theory is itself the starting point for deriving further effective theories BELOW the scale m_b

$\longrightarrow$ HQET

SCET (as applied in heavy quark physics)

Heavy Quark Effective Theory

[Reviews : Neubert, Phys. Rep. (1994)
Manohar, Wise
" Heavy Quark Physics",
Cambridge Univ. Press]

Binding due to strong interaction

$\alpha_s(\mu) \sim 1$ i.e. $\mu \approx \Lambda_{QCD}$

$|\bar{B}(p)\rangle$

$p = M_B v$

HEAVY quarks : $m_Q \gg \Lambda_{QCD}$

$p_Q = m_Q v + k \qquad [v^2 = 1]$

small "residual" momentum
$k \sim \Lambda_{QCD}$ due to interaction
with light quarks and gluons

$$\mathcal{L}_{QCD} + \mathcal{L}_{eff}^{weak}$$

$\downarrow$ perturbative matching

$$\mathcal{L}_{HQET} + \mathcal{L}'^{weak}_{eff}$$

Matrix elements of HQET
operators are m_Q - independent

HQET applies to

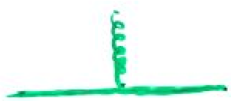
- Spectrum of heavy hadrons
(binding)

- Decays (or production ...) of heavy hadrons
provided there exists a frame in which <u>all</u>
light degrees of freedom have momenta $\ll m_Q$!

Effective propagator

$$\frac{i(\not p+m_Q)}{p^2-m_Q^2+i\varepsilon} \underset{p=m_Q v+k}{=} \frac{i\left(m_Q(1+\not v)+\overset{\text{small}}{\not k}\right)}{m_Q^2\underbrace{(v^2-1)}_{0}+2m_Q v\cdot k+\underset{\text{small}}{k^2}} = \frac{1+\not v}{2}\,\frac{i}{v\cdot k+i\varepsilon}+O\left(\frac{1}{m_Q}\right)$$

Effective vertex



$$\frac{1+\not v}{2}\, i g_s \gamma^\mu T^A\, \frac{1+\not v}{2}$$

$$= \underbrace{\frac{1+\not v}{2}\,\frac{1-\not v}{2}}_{0}\, i g_s \gamma^\mu T^A + \frac{1+\not v}{2}\, i g_s v^\mu T^A$$

$$= \frac{1+\not v}{2}\, i g_s v^\mu T^A\, \frac{1+\not v}{2}$$

$$\Rightarrow \quad \mathcal{L} = \bar Q(i\not D - m_Q)Q \longrightarrow \mathcal{L}_{eff} = \bar h_v\, i v\cdot D h_v + O\left(\frac{1}{m_Q}\right)$$

with $\not v h_v = h_v$

Leading order HQET Lagrangian

Spin-flavour symmetry

- no Dirac matrices

  invariant under spin-SU(2) $\quad h_v \to R h_v$

  $$R = e^{i\vec\varepsilon\cdot\vec S} \qquad \underset{e^i \perp v}{S^i = \tfrac{1}{2}\gamma_5 \not v \not e^i} \qquad [S^i, S^j] = i\varepsilon^{ijk} S^k$$

- no dependence on m_Q $\quad \sum_{i=1}^{N_h} \bar h_v^i\, i v\cdot D h_v^i \quad$ invariant

  under flavour-SU(N_h) $\quad h_v^i = U_{ij} h_v^j \quad$ even for $m_{Q_i} \neq m_{Q_j}$

$$\mathcal{L}^{(0)}_{HQET} = \bar{h}_v \, i v \cdot D h_v + \mathcal{L}_{light}$$

$$\mathcal{L}_{light} = -\frac{1}{4} G^2 + \sum \bar{q} (i \not{D} - m_q) q$$

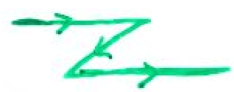
light quarks

Does not include :

1) Some $1/m_Q$ effects dropped above

2) Heavy anti-quarks

$$\frac{i(\not{p}+m_Q)}{p^2 - m_Q^2 + i\epsilon} = \frac{i(\not{p}+m_Q)}{\left(p^0 - \left[\sqrt{m_Q^2+\vec{p}^{\,2}} - i\epsilon\right]\right)\left(p^0 + \left[\sqrt{m_Q^2+\vec{p}^{\,2}} + i\epsilon\right]\right)}$$

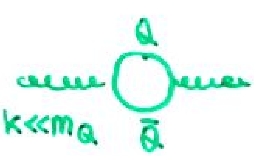


in time-ordered PT

$\uparrow$ particle pole

$\uparrow$ anti-particle pole $\rightarrow 2m_Q$

3) Heavy quark loops



4) Short-distance fluctuations are wrong

Loop integral includes region $k \sim m_Q$ where the propagator and vertex of $\mathcal{L}_{HQET}$ are plainly wrong.

! All defects concern <u>short</u> distances ($x \sim 1/m_Q$) and can be repaired by perturbative matching. Long-distance physics correctly reproduced. !

$1/m_Q$ corrections

Dim-5 operators

$$\bar{h}_v \Gamma_{\mu\nu} iD^\mu iD^\nu h_v$$

$g_{\mu\nu}, \sigma_{\mu\nu}$ $\quad$ $iv\cdot D h_v = 0$ by LO eq. of motion

$$\hookrightarrow \quad \bar{h}_v (iD_\perp)^2 h_v \,, \quad \bar{h}_v \sigma_{\mu\nu} g_s G^{\mu\nu} h_v$$

$$D_\perp^\mu = D^\mu - v\cdot D\, v^\mu$$

Derivation of the Lagrangian

fluctuate only on scale Λ_{QCD}

$$Q(x) = e^{-im_Q v\cdot x}\left(h_v(x) + H_v(x)\right)$$

i.e. $\displaystyle \begin{matrix} h_v(x) \\ H_v(x) \end{matrix} = \frac{1 \pm \not{v}}{2} e^{im_Q v\cdot x} Q(x)$

$$\begin{aligned}
\Rightarrow \quad \mathcal{L} &= \bar{Q}(i\not{D} - m_Q)Q \\
&= \bar{h}_v iv\cdot D h_v - \bar{H}_v(iv\cdot D + 2m_Q)H_v \\
&\quad + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v
\end{aligned}$$

large
H_v is (almost) non-dynamical

use e.o.m.
$(iv\cdot D + 2m_Q)H_v = i\not{D} h_v$
solve for H_v

$$= \bar{h}_v iv\cdot D h_v + \bar{h}_v i\not{D}_\perp \frac{1}{2m_Q + iv\cdot D} i\not{D}_\perp h_v$$

can be expanded in LOCAL operators

$$= \bar{h}_v iv\cdot D h_v + \bar{h}_v \frac{(iD_\perp)^2}{2m_Q} h_v + \frac{g_s}{4m_Q} \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v + \mathcal{O}(1/m_Q^2)$$

$$\not{D}_\perp \not{D}_\perp = D_\perp^2 - \frac{g_s}{2}\sigma_{\mu\nu}G^{\mu\nu}$$

$$\mathcal{L}_{HQET} = \bar{h}_v \, iv\cdot D h_v + \mathcal{L}_{light}$$

$$+ \bar{h}_v \frac{(iD_\perp)^2}{2m_Q} h_v + \frac{g_s C(\alpha_s)}{4m_Q} \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v + \ldots\ldots$$

" Kinetic energy correction "

" chromomagnetic interaction "

$\vec{\sigma}\cdot\vec{B}$ in rest frame

- Used the classical eq. of. motion. Valid only at tree - level
- Effect of loops changes coupling constants (see below). Here

$$C(\alpha_s) \overset{\overline{MS}}{=} 1 - \frac{3\alpha_s(\mu)}{2\pi}\left(\ln\frac{m_Q}{\mu} - \frac{13}{9}\right)$$

Kinetic energy not renormalized , because bilinear terms in $\bar{h}_v \ldots h_v$ must reproduce the QCD dispersion relation

$$p_Q^2 = (m_Q v + k)^2 = m_Q^2$$

- $1/m_Q$ corrections break spin - flavour symmetry but can be treated as perturbations

Work in the <u>interaction picture</u> with

$$H_0 \sim \bar{h}_v \, iv\cdot D h_v + H_{light}$$

$$H_{int} \sim 1/m_Q^n \text{ terms}$$

$\Rightarrow$ Hadron states are eigenstates of H_0 and are classified in multiplets of the spin - flavour symmetry

Evaluating hadronic matrix elements

e.g. $\langle M'_{v'} | \bar{h}_{v'} \Gamma h_v | M_v \rangle$

- completely non-perturbative
- can only exploit symmetries

$|M_v\rangle \equiv |M_v, n\rangle$ ← states of a spin-flavour symmetry multiplet

Focus on spin symmetry

$s_Q = \frac{1}{2}$ $s_\ell = -\frac{1}{2}$

$\bar{B}$ $(J=0)$ $\qquad$ $\bar{B}^*$ $(J=1, L=0)$

Wave-function of $\bar{B}, \bar{B}^*$ is unknown, but its Lorentz transformation property is that of

$$\left(u_h(v, s_Q) \bar{v}_\ell(v, s_\ell) \right)_{\alpha\beta}$$

$s_Q = \pm\frac{1}{2}$, $s_\ell = \pm\frac{1}{2}$ provides the four states of the $\bar{B}, \bar{B}^*$ multiplet

Instead of these four take linear combinations with well-defined angular momentum $J = 0, 1$

$$(\uparrow\downarrow + \downarrow\uparrow) = \frac{1+\not{v}}{2}(-\gamma_5) \qquad \bar{B}$$

$$\left. \begin{array}{c} (\uparrow\downarrow - \downarrow\uparrow) \\ \uparrow\uparrow \\ \downarrow\downarrow \end{array} \right\} = \frac{1+\not{v}}{2} \not{\varepsilon}$$

polarization vector of $\bar{B}^*$

$\bar{B}^*$

$$\varepsilon_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \qquad \varepsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

in the rest frame
else $\varepsilon \cdot v = 0$

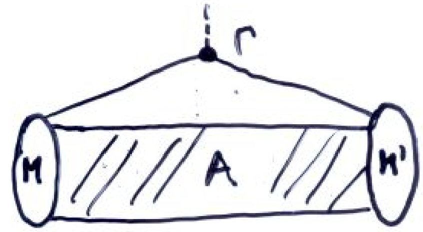
Thus

$$|M_{v},n\rangle \longleftrightarrow M(v,\epsilon_n) \equiv \frac{1+\not v}{2}\not\epsilon_n$$

with $\not\epsilon_0 \equiv -\gamma_5$ for $\bar B$

$\not\epsilon_n = \not\epsilon_\perp, \not\epsilon_3$ for $\bar B^*$

$$\langle M'_{v'}|\bar h_{v'}\Gamma h_v|M_v\rangle =$$



$$= \left[\bar M(v',\epsilon'_n)\Gamma M(v,\epsilon_n)\right]_{\alpha\beta} A(v,v')_{\beta\alpha}$$

unknown object ($\approx$ overlap of light degrees of freedom), independent of heavy quark mass and spin

Most general form (compatible with parity) $\quad A = A_1 \mathbb{1} + A_2\not v + A_3\not v' + A_4\not v\not v'$

$$\hookrightarrow \quad \mathrm{tr}\left((A_1 + A_2\not v + A_3\not v' + A_4\not v\not v')\not\epsilon'_n \frac{1+\not v'}{2}\Gamma\frac{1+\not v}{2}\not\epsilon_n\right)$$

$$= \underbrace{(A_1 - A_2 - A_3 - A_4)}_{\equiv -\xi} \mathrm{tr}\left(\bar M(v',\epsilon'_n)\Gamma M(v,\epsilon_n)\right)$$

$\frac{1+\not v}{2}\not\epsilon_n\not v = -\frac{1+\not v}{2}\not v\not\epsilon_n$

$= -\frac{1+\not v}{2}\not\epsilon_n$

$$\langle M'_{v'}|\bar h_{v'}\Gamma h_v|M_v\rangle = -\xi(v\cdot v')\ \mathrm{tr}\left(\bar M(v')\Gamma M(v)\right)$$

ONE (Isgur-Wise) form factor for all Γ and all $\quad B, B^*, D, D^* \longrightarrow B, B^*, D, D^*$

Applications of HQET

Meson masses

$$\int d^3x\, \mathcal{H}(x)$$

$$m_{H_Q} = \frac{\langle H_Q | H | H_Q \rangle}{\langle H_Q | H_Q \rangle} = m_Q + \frac{\langle M_v | H_{eff} | M_v \rangle}{\langle M_v | M_v \rangle} + \mathcal{O}(1/m_Q^2)$$

V(olume)

$$= m_Q + \langle M_v | \bar{h}_v\, iv\cdot D\, h_v | M_v \rangle$$

$$- \frac{1}{2m_Q} \langle M_v | \bar{h}_v \left[(iD_\perp)^2 + \frac{g_s}{2} \sigma_{\mu\nu} G^{\mu\nu} \right] h_v | M_v \rangle + \dots$$

(1) (2) (3)

[(1),(2)

$$\mathrm{tr}\left(A\, \bar{M}(v,\epsilon_n) M(v,\epsilon_n) \right) = (A_1 - A_2)\, \mathrm{tr}\left(\epsilon_n^* \frac{1+\not v}{2} \epsilon_n \right) \propto \mathrm{tr}(\epsilon_n^* \epsilon_n) = -4$$

$A = A_1 + A_2 \not v$ $\qquad [\epsilon_0^* = +\gamma_5]$ $\qquad$ independent of n

(3)

$$\mathrm{tr}\left(A^{\mu\nu}_{(v)}\, \bar{M}(v,\epsilon_n) \sigma_{\mu\nu} M(v,\epsilon_n) \right) = \begin{Bmatrix} -12 \\ +4 \end{Bmatrix} (A_1 - A_2) \quad \text{for} \quad \begin{matrix} P \\ V \end{matrix}$$

$A^{\mu\nu}_{(v)} = A_1 \sigma_{\mu\nu} + A_2 \sigma_{\mu\nu} \not v + A_3 (v_\mu \gamma_\nu - v_\nu \gamma_\mu)$]

$$m_{H_Q} = m_Q + \bar{\Lambda} - \frac{\lambda_1 + 3\lambda_2}{2m_Q} \quad \mathcal{O}(1/m_Q^2) \qquad B, D, \dots$$

$$m_{H_Q^*} = m_Q + \bar{\Lambda} - \frac{\lambda_1 - \lambda_2}{2m_Q} + \dots \qquad B^*, D^*, \dots$$

$$\lambda_1 \equiv \langle M_v^{(P)} | \bar{h}_v (iD_\perp)^2 h_v | M_v^{(P)} \rangle$$

$$\lambda_2 \equiv \frac{1}{6} \langle M_v^{(P)} | \bar{h}_v\, g_s \sigma G\, h_v | M_v^{(P)} \rangle$$

independent of m_Q !

- The B and D masses differ by $m_b - m_c$:

$$m_{B^{(*)}} - m_{D^{(*)}} = m_b - m_c + \mathcal{O}\left(1/m_{b,c}\right)$$

(flavour symmetry)

- H_Q and H_Q^* become degenerate as $m_Q \to \infty$

(spin symmetry)

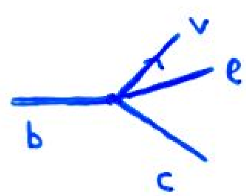
$$m_{H_Q^*} - m_{H_Q} = \frac{2\lambda_2}{m_Q} + \mathcal{O}\left(1/m_Q^2\right)$$

$$\frac{m_{B^*} - m_B}{m_{D^*} - m_D} \approx \frac{m_c}{m_b}$$

$$\frac{46\ \text{MeV}}{141\ \text{MeV}} \approx \frac{1.5\ \text{GeV}}{4.8\ \text{GeV}}$$

Semi-leptonic $\bar{B} \to D^{(*)} e\nu$ decay and $|V_{cb}|$

Effective weak interaction



$$\frac{G}{\sqrt{2}} V_{cb} (\bar{c}b)_{V-A} (\bar{e}\nu)_{V-A}$$

$(\bar{c}b)_{V-A}$ → hadronic matrix element

$(\bar{e}\nu)_{V-A}$: trivial

$$\langle D(p') | \bar{c}\gamma^\mu(1-\gamma_5) b | B(p) \rangle$$

$$= f_+(q^2)\left((p+p')^\mu - \frac{m_B^2 - m_D^2}{q^2} q^\mu \right) + f_0(q^2) \frac{m_B^2 - m_D^2}{q^2} q^\mu$$

two form factors

$$\langle D^*(p') | \bar{c}\gamma^\mu(1-\gamma_5) b | B(p) \rangle \longrightarrow A_0(q^2), A_1(q^2), A_2(q^2), V(q^2)$$

HQET

$$\langle D^*(p') | (\bar{c} b)_{V-A} | B(p) \rangle \overset{\text{leading order}}{\propto} \langle M_{v'} | \bar{h}_{v'} \Gamma h_v | M_v \rangle$$

$\longrightarrow$ Isgur-Wise form factor $\xi(v \cdot v')$

- $f_{+,0,T}$, $A_{0,1,2}$, V, $T_{1,2}$ (10 unknown form factors) $\longrightarrow$ $\xi(v \cdot v')$ up to $1/m_{b,c}$ corrections

- $\xi(v \cdot v')$ is known at one point : $\xi(1) = 1$

$$\langle M_v^{(P)} | \bar{h}_v \gamma^\mu h_v | M_v^{(P)} \rangle = 2\xi(1) v^\mu$$

[from $\mathrm{tr}(\bar{M}_v^{(P)} \gamma^\mu M_v^{(P)})$]

$$N = \int d^3\vec{x}\, \bar{h}_v \gamma^0 h_v$$ = heavy quark number operator

$$\Rightarrow \quad 2\xi(1) v^0 \cdot V = \langle M_v^{(P)} | N | M_v^{(P)} \rangle$$

$$= \langle M_v^{(P)} | M_v^{(P)} \rangle = 2 v^{(0)} \cdot V$$

[$\langle M_{v'} | M_v \rangle = 2 v^0 (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$]

$$\Rightarrow \quad \xi(1) = 1$$

Kinematics

$$w \equiv v \cdot v' = \frac{p \cdot p'}{m_B m_D} = \frac{m_B^2 + m_D^2 - q^2}{2 m_B m_D} \qquad 0 \le q^2 = (p-p')^2 \le (m_B - m_D)^2$$

momentum transfer

In B rest frame the boost of the D meson is

$$\gamma = \frac{E_D}{m_D} = v \cdot v' \le \frac{m_B^2 + m_D^2}{2 m_B m_D} \approx 1.6 \ll \frac{m_Q}{\Lambda_{QCD}}$$

$\Rightarrow$ typical momenta of light degrees of freedom are still $O(\Lambda_{QCD})$ and HQET does apply

$$\frac{d\Gamma(\bar{B} \to D^* \ell \nu)}{dw} = |V_{cb}|^2 \, r_*(w) \sqrt{w^2 - 1} \; F_*^2(w)$$

$$\frac{d\Gamma(B \to D \ell \nu)}{dw} = |V_{cb}|^2 \, r(w) \, (w^2 - 1)^{3/2} \, F^2(w)$$

harder to measure for $w \to 1$

data for $w \to 1$

known kinematic & phase space factors

$F_{(*)}(w) \xrightarrow[w \to 1]{} 1$

(heavy quark limit)

$\Rightarrow$ Measurement of $|V_{cb}|$

3% exp. error

Data: (HFAG)

$$|V_{cb}| \, F_*(1) = (37.6 \pm 0.9) \times 10^{-3}$$

$$|V_{cb}| \, F(1) = (42.2 \pm 3.7) \times 10^{-3}$$

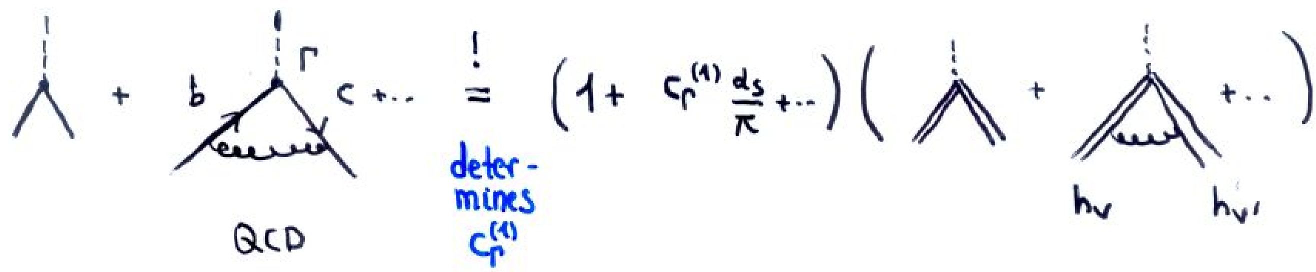
$F_{(*)}(1) = 1 + ??$

Leading order not enough for precise $|V_{cb}|$ determination

$$\bar{c}\,\Gamma\, b = C_\Gamma(\alpha_s)\, \bar{h}_{v'}\Gamma h_v + \sum_i D_i(\alpha_s)/m_Q \; J^i(1/m) + \dots$$

compensates for incorrect short distance behaviour

$1/m_Q$ effects



$$\hookrightarrow \quad F_*(1) = \eta_A(\alpha_s) + \mathcal{O}\left(1/m_{c,b}^2\right) \quad ! \quad \text{at } w = 1$$

$$F(1) = \eta_V(\alpha_s) + \mathcal{O}\left(1/m_{c,b}\right)$$

$$\eta_A = 1 + \frac{\alpha_s}{\pi}\left(\frac{m_b + m_c}{m_b - m_c}\ln\frac{m_b}{m_c} - \frac{8}{3}\right) + \mathcal{O}(\alpha_s^2)$$

$$\approx 0.96$$

$$F_*(1) = 0.91 \pm 0.05$$

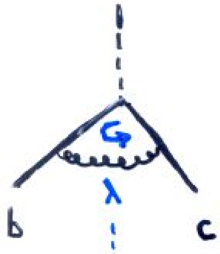
error from model-dep. power correction

$1/m_c^2$ correction from lattice ; not small

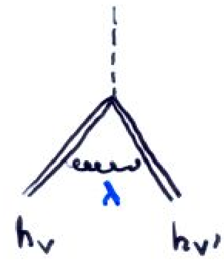
$\Rightarrow \quad \approx 5\%$ error on $|V_{cb}|$

Radiative corrections, momentum regions and factorisation

Work at leading order in $1/m_b$ for illustration



$$= C_\Gamma^{(1)}(m_Q,\mu)\,\frac{\alpha_s}{\pi} \;+$$



gluon mass to regulate IR divergence

depends only on λ

depends on $(v\cdot v' = 1)$

m_b, m_c – short-distance scales

λ – long-distance scale

loop momentum

$k \sim m_Q$ "hard"

$k \sim \lambda$ "soft"

In some well-defined sense the integral is the sum

"hard + soft"

see below

useful for SCET

Feynman rules in HQET reproduce the soft contribution

$C_\Gamma^{(1)}$ is independent of λ

$\simeq$ "hard"

EFT achieves FACTORIZAION of effects related to the two scales

HQET : summary

* EFT for heavy quarks (near mass-shell) interacting with soft quarks / gluons (momentum $\sim \Lambda_{QCD} \ll m_Q$)

* Systematic expansion of matrix elements in Λ_{QCD}/m_b

  - calculable matching coefficients
  - hadronic matrix elements of B, B^*, D, D^* (and other multiplets) related by spin-flavour symmetry

* Factorization of short- and long-distances

* Most important phenomenological result: $|V_{cb}|$ from $B \to D^* \ell \nu$

* HQET representation of heavy quarks also used in SCET