III. Supersymmetry – extending Poincaré symmetry

Initially, attempts were made to extend Poincaré symmetry by combining space-time and internal symmetries in a non-trivial way. Coleman and Mandula proved that these attempts would always fail. They assumed all symmetry generators satisfy commutation relations.

The breakthrough was achieved by considering a new class of "fermionic" generators that satisfy anti-commutation relations. Haag, Lopuszanski and Sohnius discovered which fermionic symmetry generators were allowed and proved they must transform either as $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$, i.e. spin-$\frac{1}{2}$ generators.

The above result implies that:

$$[Q_\alpha, J^{\mu\nu}] = i\sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta$$

where Q_α is a $(\frac{1}{2},0)$ symmetry operator. In addition,

$$[Q_\alpha, P^\mu] = 0$$

since Q_α is translationally invariant (no explicit x-dependence).

Similarly, $\overline{Q}^{\dot\alpha}$ is a $(0,\frac{1}{2})$ symmetry operator, so

$$[\overline{Q}^{\dot\alpha}, J^{\mu\nu}] = i\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta}$$

$$[\overline{Q}^{\dot\alpha}, P^\mu] = 0$$

The Q's satisfy anti-commutation relations. Given the Lorentz properties of the Q's, there is little freedom left. For example, $\{Q_\alpha, \overline{Q}_{\dot\beta}\}$ transforms as $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ which is a four-vector.

On the other hand,

$\{Q_\alpha, Q_\beta\}$ transforms as $(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0) = (1,0)\oplus(0,0)$

$\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$ transforms as $(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2}) = (0,1)\oplus(0,0)$

There is no generator that transforms as $(0,0)$, so at best:

$$\{Q_\alpha, Q^\beta\} = s\,\sigma^{\mu\nu}{}_\alpha{}^\beta J_{\mu\nu}$$

$$\{\bar{Q}^{\dot\alpha}, \bar{Q}_{\dot\beta}\} = s^* \bar{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} J_{\mu\nu}$$

for some complex number s.

But, since $[Q_\alpha, P^\mu] = [\bar{Q}_{\dot\alpha}, P_\mu] = 0$ and $[J_{\mu\nu}, P^\lambda] \neq 0$, it follows that $s=0$.

Since $\{Q_\alpha, \bar{Q}_{\dot\beta}\}$ transforms as a four vector,

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = t\,\sigma^\mu_{\alpha\dot\beta} P_\mu$$

for some complex number t.

From the identity $\sigma^\mu_{\alpha\dot\beta}\,\bar{\sigma}^{\nu\dot\beta\alpha} = 2g^{\mu\nu}$,

$$\bar{\sigma}_\mu{}^{\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2t P_\mu$$

For $\mu=0$, $\bar{\sigma}_0 = 1$, so

$$2tP_0 = Q_1 Q_1^* + Q_1^* Q_1 + Q_2 Q_2^* + Q_2^* Q_2 \geq 0$$

Since $P_0 \geq m \geq 0$ for physical states, it follows that $t > 0$.

<u>Convention:</u> $t=2$ (can be achieved by rescaling the Q_α).

$$\boxed{\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu}$$

where the 2 is conventional (and can be achieved by re-scaling the Q's). The other anti-commutators are:

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

<u>exercise</u>: In 4-component notation, define $Q_M = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix}$. Show that:

$$[Q_M, J^{\mu\nu}] = \tfrac{1}{2}\sigma^{\mu\nu} Q_M$$
$$\{Q_M, \bar{Q}_M\} = 2\gamma^\mu P_\mu \qquad \text{where } \bar{Q}_M = Q_M^\dagger A.$$

<u>Theorem</u>:

The vanishing of the vacuum energy is a necessary and sufficient condition for the existence of a unique supersymmetric vacuum.

<u>proof</u>: multiply $\{Q_\alpha, \bar{Q}_{\dot\beta}\}$ by $\bar\sigma^{\nu\dot\beta\alpha}$ and use $\operatorname{tr}\sigma^\mu\bar\sigma^\nu = 2g^{\mu\nu}$ to obtain:

$$\bar\sigma^{\mu\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 4P^\mu$$

For $\mu = 0$, this reads:

$$4P^0 = Q_1 Q_1^* + Q_1^* Q_1 + Q_2 Q_2^* + Q_2^* Q_2$$

Thus, if $|0\rangle$ is the vacuum state, then

$$\langle 0|P^0|0\rangle \geq 0$$

and

$$\langle 0|P^0|0\rangle = 0 \iff Q_\alpha|0\rangle = 0.$$

if this condition is satisfied then the vacuum is supersymmetric

Recall the Casimir operators of the Poincaré algebra, P^2 and w^2.

Note that:

$$[P^2, Q_\alpha] = [P^2, \overline{Q}_{\dot\alpha}] = 0$$

but,

$$[w^2, Q_\alpha] \neq 0$$
$$[w^2, \overline{Q}_{\dot\alpha}] \neq 0$$

Thus, the irreducible representations of the supersymmetry algebra will contain different spins. If $|B\rangle$ is a boson and $|F\rangle$ is a fermion, then,

$$Q_\alpha |B\rangle = |F\rangle \qquad \text{(schematic)}$$
$$Q_\alpha |F\rangle = |B\rangle$$

where $|F\rangle$ and $|B\rangle$ differ by half a unit of spin, since Q_α is a $(\frac{1}{2}, 0)$ symmetry operator.

<u>Definition:</u> $(-1)^F$ is an operator defined such that

$$(-1)^F |B\rangle = +|B\rangle$$
$$(-1)^F |F\rangle = -|F\rangle$$

<u>Theorem:</u>

(i) $Q_\alpha (-1)^F = -(-1)^F Q_\alpha$

(ii) $\mathrm{tr}\,(-1)^F = 0$ (for fixed non-zero P_μ)

<u>consequence of (ii)</u>: supersymmetric multiplets contain equal numbers of bosonic and fermionic degrees of freedom

proof of (ii): evaluate $\mathrm{tr}\,[(-1)^F \{Q_\alpha, \overline{Q}_{\dot\beta}\}]$ in two ways.

First way - use the anticommutation relation for the Q's.

Second way - expand the anticommutator and manipulate the expression using (i) until you get zero.

Casimir operators of the supersymmetric algebra

P^2 remains a Casimir operator since it commutes with all the generators. This means that all the states that make up a supersymmetric multiplet have the same mass.

W^2 is no longer a Casimir operator. We need a supersymmetric generalization of the Pauli-Lubanski vector W_μ.

definition:

$$B_\mu \equiv W_\mu - \frac{1}{8}\sigma_\mu^{\dot\alpha\beta}[Q_\beta, \bar{Q}_{\dot\alpha}]$$

note:

$[B_\mu, B_\nu] = -i\varepsilon_{\mu\nu\rho\sigma}B^\rho P^\sigma$

$[B_\mu, P_\nu] = 0$

The factor of $\frac{1}{8}$ was chosen so that

$$[B_\mu, Q_\alpha] = -\frac{1}{2}P_\mu Q_\alpha$$
$$[B_\mu, \bar{Q}_{\dot\alpha}] = \frac{1}{2}P_\mu \bar{Q}_{\dot\alpha}$$

Next, we define:

$$C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu$$

It is easy to check that

$$[C_{\mu\nu}, Q_\alpha] = [C_{\mu\nu}, \bar{Q}_{\dot\alpha}] = [C_{\mu\nu}, P_\sigma] = 0.$$

Since $C_{\mu\nu}C^{\mu\nu}$ is Lorentz invariant, it follows that

$$[C_{\mu\nu}C^{\mu\nu}, J^{\alpha\beta}] = 0.$$

Hence, $C_{\mu\nu}C^{\mu\nu}$ is the second Casimir operator of the supersymmetry algebra.

We can write:

$$C_{\mu\nu}C^{\mu\nu} = 2[B^\mu B_\mu P^2 - (B^\mu P_\mu)^2]$$

Case 1: $P^2 > 0$.

In a frame where $P^\mu = (m; \vec{0})$,

$$C^{\mu\nu} C_{\mu\nu} = 2m^2 (B^\mu B_\mu - B_0^2) = -2m^2 |\vec{B}|^2$$

Moreover, in this frame:

$$[B^i, B^j] = i m \varepsilon^{ijk} B^k$$

Define $m J^k = B^k$. Then the J^k satisfy angular momentum commutation relations and:

$$C^{\mu\nu} C_{\mu\nu} = -2m^4 \vec{J}^2$$

has eigenvalues $-2m^4 j(j+1)$, where $j = 0, \frac{1}{2}, 1, \ldots$ is the "superspin". The irreducible representations are thus $|m, j\rangle$.

Using $\vec{w} = m \vec{S}$,

$$J^k = S^k - \frac{1}{8m} \sigma^{k\dot{\alpha}\beta} [Q_\beta, \bar{Q}_{\dot{\alpha}}]$$

In the rest frame, $\{Q_1, \bar{Q}_{\dot{1}}\} = \{Q_2, \bar{Q}_{\dot{2}}\} = 2m$ and all other anti-commutators vanish. Thus, the state $|\Omega\rangle \equiv Q_1 Q_2 |m, j, j_3\rangle$ satisfies the condition $Q_\alpha |\Omega\rangle = 0$.

The only non-vanishing states then are:

$$|\Omega\rangle,\ \bar{Q}^{\dot{1}}|\Omega\rangle,\ \bar{Q}^{\dot{2}}|\Omega\rangle \text{ and } \bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle$$

We can compute the spin of these states by working out the eigenvalues with respect to S^3 and $\vec{S}^2$.

$$|m, j\rangle \quad \begin{cases} j_3 = j \\ j_3 = j-1 \\ \vdots \\ j_3 = -j \end{cases} \quad \xrightarrow{\text{fixed } j_3} \quad \begin{array}{ll} S_3 = j_3 & |\Omega\rangle \\ S_3 = j_3 + \frac{1}{2} & \bar{Q}^{\dot{1}}|\Omega\rangle \\ S_3 = j_3 - \frac{1}{2} & \bar{Q}^{\dot{2}}|\Omega\rangle \\ S_3 = j_3 & \bar{Q}^{\dot{1}}\bar{Q}^{\dot{2}}|\Omega\rangle \end{array}$$

example: $j = 0$

possible values of s_3:

$$s_3 = 0, +\frac{1}{2}, -\frac{1}{2}, 0$$

corresponds to two real scalars* and one Majorana fermion.

scalar degrees of freedom = 2 [*equivalently, one complex scalar]

fermion degrees of freedom = 2

exercise: show that $j = 1/2$ corresponds to a real vector field, a real scalar field and two Majorana fermions (or equivalently one Dirac fermion).

Case 2: $p^2 = 0$

One can show that for $P^2 = 0$,

$$L_\mu \equiv W_\mu - \frac{1}{16} \sigma_\mu^{\dot{\alpha}\beta} [Q_\beta, \bar{Q}_{\dot{\alpha}}]$$

satisfies:
$$P^\mu L_\mu = 0$$
$$[L_\mu, L_\nu] = -i \epsilon_{\mu\nu\alpha\beta} L^\alpha P^\beta$$

(the same relations satisfied by W_μ). Thus, for $P^2 = 0$, L_μ is proportional to P_μ.

$$L_\mu = (K + \tfrac{1}{4}) P_\mu \qquad K = \text{super-helicity}$$

If $Q_\alpha |\Omega\rangle$ as before, a simple computation yields:

$$W_\mu |\Omega\rangle = (K + \tfrac{1}{2}) P_\mu |\Omega\rangle$$

However, for $P^2 = 0$, only two states survive. The massless supermultiplet consists of particles of helicity K and $K + \frac{1}{2}$. We must add the corresponding anti-particles (CPT-conjugates), which yields states of helicity $-K$ and $-(K + \frac{1}{2})$.

examples:

(i) $K=0$

helicities $0, \frac{1}{2}$ $\oplus$ helicities $0, -\frac{1}{2}$

corresponding to a massless complex scalar and a massless Majorana fermion

(ii) $K=\frac{1}{2}$

helicities $\frac{1}{2}, 1$ $\oplus$ helicities $-\frac{1}{2}, -1$

corresponding to a massless Majorana fermion and a massless real vector field.

Further generalizations

Extended supersymmetry introduces N spin-1/2 $(\frac{1}{2},0)$ generators, Q_α^A $(A=1,\dots,N)$ and N $(0,\frac{1}{2})$ generators $\overline{Q}_{\dot{\alpha}A}$. The algebra involving the Q's is more complicated.

In these lectures, we will not make this additional generalization. The theories discussed here are based on $N=1$ supersymmetry.

The main reason for this choice is due to the fact that $N=1$ theories can describe chiral fermions (as seen in nature). For $N>1$, all left-handed fermions have right-handed partners, so the construction of realistic theories of this type are much more difficult.

R-invariance

The supersymmetry algebra can be extended slightly by noting that Q_α is inherently complex. The supersymmetric algebra is unchanged under:

$$Q_\alpha \rightarrow e^{-i\gamma} Q_\alpha$$

$$\bar{Q}_{\dot{\alpha}} \rightarrow e^{i\gamma} \bar{Q}_{\dot{\alpha}}$$

where γ is a real number. Thus, introduce a new symmetry generator R such that:

$$e^{i\gamma R} Q_\alpha e^{-i\gamma R} = e^{-i\gamma} Q_\alpha$$

$$e^{i\gamma R} \bar{Q}_{\dot{\alpha}} e^{-i\gamma R} = e^{i\gamma} \bar{Q}_{\dot{\alpha}}$$

Taking γ infinitesimal,

$$[R, Q_\alpha] = -Q_\alpha$$

$$[R, \bar{Q}_{\dot{\alpha}}] = \bar{Q}_{\dot{\alpha}}$$

We may choose $[R, P^\mu] = [R, J^{\mu\nu}] = 0$ to extend the supersymmetry algebra.

IV. Supersymmetric theories of spin-½ fermions and their spin-0 boson superpartners

We have seen that the simplest supermultiplet consists of two real scalars (or equivalently, a complex scalar) and a Majorana fermion, all with mass m.
(corresponding to superspin $j=0$ if massive, or to superhelicity 0 if massless).

A Lagrangian that respects the supersymmetry algebra is given by:

$$\mathcal{L} = (\partial_\mu A)^*(\partial^\mu A) + i\bar{\Psi}\bar{\sigma}^\mu\partial_\mu\Psi - \left|\frac{dW}{dA}\right|^2 - \frac{1}{2}\left[\frac{d^2W}{dA^2}\Psi\Psi + \left(\frac{d^2W}{dA^2}\right)^*\bar{\Psi}\bar{\Psi}\right]$$

where $W = W(A)$ is an arbitrary holomorphic function of A (i.e. W is an analytic function that depends on A but not A^*)

A = complex scalar
Ψ = two-component Majorana fermion

If $W(A)$ is a cubic polynomial in A, then this is a renormalizable QFT called the Wess-Zumino model.

Simple example: $W = \frac{1}{2}mA^2$

result: a free theory of a complex scalar and a Majorana fermion, both of mass m.

a renormalizable example with interactions

$$W = \tfrac{1}{2} m A^2 + \tfrac{1}{3} g A^3 \qquad\qquad m, g \text{ real}$$

$$\mathcal{L} = (\partial_\mu A)^*(\partial^\mu A) + i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \tfrac{1}{2} m(\psi\psi + \bar{\psi}\bar{\psi}) - m^2 A^* A$$
$$- g(A\psi\psi + A^*\bar{\psi}\bar{\psi}) - mg A^*A(A + A^*) - g^2(A^*A)^2.$$

note that:

(i) $m_A = m_\psi$

(ii) $g_{A\psi\psi} = g$ while $g_{(A^*A)^2} = g^2$

Both (i) and (ii) are consequences of supersymmetry. Thus, supersymmetry relates different couplings of the theory.

Convert to four-component notation. Define $A = \frac{1}{\sqrt{2}}(S + iP)$. Then,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu S)^2 + \tfrac{1}{2}(\partial_\mu P)^2 - \tfrac{1}{2} m^2 (S^2 + P^2)$$
$$+ \tfrac{i}{2}\bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - \tfrac{1}{2} m \bar{\Psi}_M \Psi_M$$
$$- \frac{g}{\sqrt{2}}\left[S\bar{\Psi}_M \Psi_M - iP\bar{\Psi}_M \gamma_5 \Psi_M \right]$$
$$- \frac{mg}{\sqrt{2}} S(S^2 + P^2) - \tfrac{1}{4} g^2 (S^2 + P^2)^2.$$

Note that S is a scalar and P is a pseudoscalar.

[Had we chosen g complex, we would have generated CP-violating scalar-fermion interactions.]

The supersymmetric transformations are:

$$\delta_\xi A = \sqrt{2}\, \xi^\alpha \psi_\alpha$$
$$\delta_\xi \psi_\alpha = -i\sqrt{2}\, \sigma^\mu_{\alpha\dot\beta}\, \bar\xi^{\dot\beta}\, \partial_\mu A - \sqrt{2}\, \xi_\alpha \left(\frac{dW}{dA}\right)^*$$

where ξ is a constant (i.e. x-independent) anti-commuting infinitesimal parameter.

Note: by hermitian conjugation, one also obtains:

$$\delta_\xi A^* = \sqrt{2}\, \bar\xi_{\dot\alpha}\, \bar\psi^{\dot\alpha}$$
$$\delta_\xi \bar\psi_{\dot\alpha} = i\sqrt{2}\, \xi^\beta \sigma^\mu_{\beta\dot\alpha}\, \partial_\mu A^* - \sqrt{2}\, \bar\xi_{\dot\alpha} \frac{dW}{dA}$$

One can check that

$$\delta_\xi \mathcal{L} = \partial_\mu K^\mu$$ (exercise: derive K^μ explicitly)

which means that the action, $\int d^4x\, \mathcal{L}$, is invariant, if the field equations are satisfied.

Indeed, this is a symmetry. But is it supersymmetry?

First, consider ordinary space-time translations:

$$e^{ia_\mu P^\mu} \phi(x)\, e^{-ia_\mu P^\mu} = \phi(x+a)$$

for infinitesimal a,

$$i[P^\mu, \phi(x)] = \partial^\mu \phi(x)$$

Here, $\phi(x)$ is any generic field, either A or ψ.

Thus, for an infinitesimal space-time translation,

$$\delta_a \phi(x) \equiv \phi(x+a) - \phi(x)$$
$$\simeq a^\mu \partial_\mu \phi(x)$$
$$= i a^\mu [P_\mu, \phi(x)]$$

Likewise, if $\delta_\xi \phi(x)$ is a supersymmetric transformation, we expect:

$$\delta_\xi \phi(x) = i[\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha} \bar{Q}^{\dot\alpha}, \phi(x)]$$

Consider:

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi(x) = \left[i(\eta Q + \bar\eta \bar Q), \left[i(\xi Q + \bar\xi \bar Q), \phi(x) \right] \right] - (\xi \leftrightarrow \eta)$$

$$= \left[\left[i(\eta Q + \bar\eta \bar Q), i(\xi Q + \bar\xi \bar Q) \right], \phi(x) \right]$$

using the Jacobi identity.

Finally, use the anti-commutation relations of the Q's. Since ξ and η are anti-commuting numbers, we have e.g.,

$$[\eta Q, \bar\xi \bar Q] = 2\eta^\alpha \sigma^\mu_{\alpha\dot\beta} \bar\xi^{\dot\beta} P_\mu$$

Thus, one ends up with:

$$[\delta_\eta, \delta_\xi]\phi(x) = 2(\xi \sigma^\mu \bar\eta - \eta \sigma^\mu \bar\xi)[P_\mu, \phi(x)]$$
$$= -2i(\xi \sigma^\mu \bar\eta - \eta \sigma^\mu \bar\xi)\partial_\mu \phi(x)$$

Let us test this result, using

$$\delta_\xi A = \sqrt{2}\,\xi^\alpha \psi_\alpha$$
$$\delta_\xi \psi_\alpha = -i\sqrt{2}\,\sigma^\mu_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}\,\partial_\mu A - \sqrt{2}\,\xi_\alpha \left(\frac{dW}{dA}\right)^*$$

The result:

(i) $[\delta_\eta \delta_\xi - \delta_\xi \delta_\eta] A(x) = -2i(\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi)\partial_\mu A$

(ii) $[\delta_\eta \delta_\xi - \delta_\xi \delta_\eta] \psi_\alpha(x) = -2i(\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi)\partial_\mu \psi + R$

where $R=0$ if I impose the field equations satisfied by ψ.

We say that the supersymmetry algebra is <u>realized</u> <u>on-shell</u>, i.e. after imposition of the field equations.

The derivation of (ii) is non-trivial, and requires among other things the use of Fierz identities. These are a little simpler for two-component fermions as compared to four-component fermions. All such identities are based on:

$$\delta_\alpha{}^\beta \delta^{\dot\rho}{}_{\dot\gamma} = \tfrac{1}{2}\sigma^\mu_{\alpha\dot\gamma}\,\bar\sigma_\mu^{\dot\rho\beta}$$
$$\delta_\alpha{}^\beta \delta_\rho{}^\gamma = \tfrac{1}{2}\left[\delta_\alpha{}^\gamma \delta_\rho{}^\beta - \sigma^{\mu\nu}{}_\alpha{}^\gamma\,\sigma_{\mu\nu\rho}{}^\beta\right]$$
$$\delta^{\dot\beta}{}_{\dot\alpha}\,\delta^{\dot\rho}{}_{\dot\gamma} = \tfrac{1}{2}\left[\delta^{\dot\beta}{}_{\dot\gamma}\,\delta^{\dot\rho}{}_{\dot\alpha} - \bar\sigma^{\mu\nu\dot\beta}{}_{\dot\gamma}\,\bar\sigma_{\mu\nu}{}^{\dot\rho}{}_{\dot\alpha}\right]$$

which follow from the completeness of $\{I, \vec\sigma\}$ over the set of 2×2 matrices.

An alternative approach: Noether's theorem

By Noether's theorem, an invariance of the action implies the existence of a conserved current.

Given δ_ξ as defined above, we found that $\delta_\xi \mathcal{L} = \partial_\mu K^\mu$ for some combination of fields K^μ. Then, the Noether supercurrent corresponding to this invariance is:

$$\xi^\alpha J^\mu_\alpha + \bar{\xi}_{\dot\alpha} \bar{J}^{\mu\dot\alpha} = \sum_X \delta_\xi X \frac{\delta \mathcal{L}}{\delta(\partial_\mu X)} - K^\mu$$

where we sum over $X = A, \psi$. [exercise: evaluate J^μ_α explicitly in the Wess-Zumino model.]

Note that J^μ_α has both a vector and spinor index.

Noether's theorem states that, after imposing the field equations,

$$\partial_\mu J^\mu_\alpha = \partial_\mu \bar{J}^{\mu\dot\alpha} = 0$$

i.e. the supercurrent is conserved. The supercharges are defined in the usual way:

$$Q_\alpha = \int d^3x\, J^0_\alpha \,, \qquad \bar{Q}^{\dot\alpha} = \int d^3x\, \bar{J}^{0\dot\alpha}$$

exercise: using the canonical commutation relations satisfied by the bosonic field A, and the canonical anti-commutation relations satisfied by ψ, show that:

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

where P^μ is expressed in terms of the quantum fields.*

*P^μ is the Noether charge of space-time translations

Auxiliary fields

The supersymmetric transformations shown are not optimal.
Note that:

(i) the transformations are non-linear if W is not quadratic in A.

(ii) the supersymmetry algebra is realized only on-shell.

We avoid both (i) and (ii) by introducing an auxiliary complex scalar field F.

Consider the alternative Lagrangian:

$$\mathcal{L} = (\partial_\mu A)^*(\partial^\mu A) + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^*F$$

$$-F\frac{dW}{dA} - F^*\left(\frac{dW}{dA}\right)^* - \frac{1}{2}\left[\frac{d^2W}{dA^2}\psi\psi + \left(\frac{d^2W}{dA^2}\right)^*\bar{\psi}\bar{\psi}\right]$$

F is an auxiliary field since $\mathcal{L}$ depends on F but not $\partial_\mu F$.
So, F has trivial dynamics. The field equations for F and F^* are:

$$\frac{\partial\mathcal{L}}{\partial F} = 0 \quad , \quad \frac{\partial\mathcal{L}}{\partial F^*} = 0$$

That is,

$$F^* = \frac{dW}{dA} \quad , \quad F = \left(\frac{dW}{dA}\right)^* .$$

Inserting these equations back into $\mathcal{L}$ returns us to the original Lagrangian. The theories are identical.

But, consider $\mathcal{L}$ before solving for F and F^*, and examine the following supersymmetric transformation:

$$\delta_\xi A = \sqrt{2}\,\xi^\alpha \psi_\alpha$$
$$\delta_\xi \psi_\alpha = -i\sqrt{2}\,\sigma^\mu_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}\,\partial_\mu A - \sqrt{2}\,\xi_\alpha F$$
$$\delta_\xi F = -i\sqrt{2}\,\partial_\mu \psi^\alpha \sigma^\mu_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}$$

One can now check the following results:

(i) $\delta_\xi \mathcal{L} = \partial_\mu \tilde{K}^\mu$ [exercise: find the explicit expression for $\tilde{K}^\mu$]

without imposing the field equations.

(ii) $[\delta_\eta \delta_\xi - \delta_\xi \delta_\eta]\, X(x) = -2i(\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi)\,\partial_\mu X$

for $X = A, \psi$, and F without imposing the field equations.

We say that the supersymmetry algebra is realized off-shell

Remarks:

1. Note the mass dimensions of the fields:
   $[\phi] = 1$, $[\psi] = \frac{3}{2}$, $[F] = 2$, which implies that $[\xi] = -\frac{1}{2}$.

2. Since $\delta_\xi F$ is a total divergence, $\int d^4x\, F$ is invariant under the supersymmetric transformation. This will be useful later.

note: $\delta_\xi F$ is a total divergence is a consequence of dimensional analysis. Since the supersymmetric transformation law is linear in the fields, $\delta_\xi F$ must involve ∂_μ since $[\partial_\mu] = 1$.

Counting degrees of freedom

On shell counting:

complex scalar A	2
Majorana spinor ψ	2

Before imposing the field equations, ψ_α $(\alpha=1,2)$ has four degrees of freedom, since ψ is complex. [Equivalently, count ψ_α and $\bar{\psi}_{\dot\alpha}$ as four independent degrees of freedom].

The field equations are:

$$i\bar{\sigma}^\mu \partial_\mu \psi = \left(\frac{d^2W}{dA^2}\right)^* \bar{\psi}$$

and these relate ψ and $\bar{\psi}$, eliminating two of four degrees of freedom. [If $\frac{d^2W}{dA^2}=0$, then $i\bar{\sigma}^\mu\partial_\mu\psi=0$ is a relation between ψ_1 and ψ_2]

Note: both A and ψ satisfy Klein-Gordon type field equations as well, but these do not affect the counting.

Off shell counting:

complex scalar A	2
Majorana spinor ψ_α, $\bar{\psi}_{\dot\alpha}$	4
complex auxiliary field F	2

totals:

boson degrees of freedom = 4

fermion degrees of freedom = 4

In both cases, supersymmetry guarantees the equality of the number of boson and fermion degrees of freedom.

Lessons from the Wess-Zumino model

1. It is not clear how to build supersymmetric Lagrangians starting with a known supermultiplet of particles.

2. The supersymmetric transformation laws are not immediately obvious, even if a supersymmetric Lagrangian is given.

3. Checking that the supersymmetric transformations satisfy the supersymmetric algebra is quite laborious.

4. Checking that the action is invariant under the supersymmetric transformation is also tedious.

5. Off-shell supersymmetry increases the required number of fields, but leads to supersymmetry transformation laws that are linear in the fields.

6. $\int F(x)\, d^4x$ is invariant under the supersymmetry transformation. [a hint for (1) above?]

Our goal: develop a formalism in which step (1) is trivial and steps (2)–(4) are automatic.

Superfields and superspace

Recall that our quantum fields satisfy:

$$\phi(x) = e^{ix\cdot P}\phi(0)e^{-ix\cdot P}$$

where $\quad i[P^\mu, \phi(x)] = \partial^\mu \phi$

and $\quad \delta_a \phi(x) \equiv \phi(x+a) - \phi(x) = a^\mu \partial_\mu \phi(x) = i a^\mu [P_\mu, \phi(x)]$.

We then defined the supersymmetric transformation law in analogy with δ_a

$$\delta_\xi \phi(x) = i[\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha}, \phi(x)].$$

Here, $\phi(x)$ can be any field: A, ψ, F, etc.

It looks like Q_α and $\bar{Q}^{\dot\alpha}$ are generating a translation by a non-commuting infinitesimal ξ_α and $\bar{\xi}^{\dot\alpha}$. But what is being translated?

Let us extend space-time by introducing non-commuting co-ordinates:

$$\theta^\alpha, \ \bar{\theta}_{\dot\alpha} \qquad\qquad \alpha, \dot\alpha = 1, 2$$

i.e., $\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}_{\dot\alpha}, \bar{\theta}_{\dot\beta}\} = \{\theta^\alpha, \bar{\theta}_{\dot\beta}\} = 0$.

Extend the translation operator $e^{ix\cdot P}$ to the super-translation operator:

$$G(x^\mu, \theta, \bar\theta) = e^{i(x\cdot P + \theta Q + \bar\theta \bar Q)}$$

<u>Jargon</u>: $(x^\mu; \theta^\alpha; \bar\theta_{\dot\alpha})$ is an 8-dimensional <u>superspace</u>.

Extend the field operator, $\phi(x) = e^{ix\cdot P}\phi(0)e^{-ix\cdot P}$

to the superfield:

$$\phi(x,\theta,\bar{\theta}) = G(x,\theta,\bar{\theta})\,\phi(0,0,0)\,G^{-1}(x,\theta,\bar{\theta})$$

<u>exercise:</u> Prove that:

$$G(y,\xi,\bar{\xi})G(x,\theta,\bar{\theta}) = G(x+y+i(\xi\sigma\bar{\theta}-\theta\sigma\bar{\xi}),\ \xi+\theta,\ \bar{\xi}+\bar{\theta})$$

note this non-trivial
extra space-time
translation

[Hint: use the BCH formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}\ ]$$

Consider

$$G(y,\xi,\bar{\xi})\phi(x,\theta,\bar{\theta})G^{-1}(y,\xi,\bar{\xi})$$

$$= G(y,\xi,\bar{\xi})G(x,\theta,\bar{\theta})\phi(0,0,0)[G(y,\xi,\bar{\xi})G(x,\theta,\bar{\theta})]^{-1}$$

$$= \phi(x+y+i(\xi\sigma\bar{\theta}-\theta\sigma\bar{\xi}),\ \xi+\theta,\ \bar{\xi}+\bar{\theta})$$

For infinitesimal y, ξ and $\bar{\xi}$, we can write:

$$G(y,\xi,\bar{\xi}) \simeq 1 + i(y\cdot P + \xi Q + \bar{\xi}\bar{Q})$$

and Taylor expand

$$\phi(x+y+i(\xi\sigma\bar{\theta}-\theta\sigma\bar{\xi}),\ \xi+\theta,\ \bar{\xi}+\bar{\theta})$$

$$\simeq \phi(x,\theta,\bar{\theta}) + [y^\mu + i(\xi\sigma^\mu\bar{\theta}-\theta\sigma^\mu\bar{\xi})]\partial_\mu\phi(x,\theta,\bar{\theta})$$

$$+\left(\xi^\alpha\frac{\partial}{\partial\theta^\alpha} + \bar{\xi}_{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\right)\phi(x,\theta,\bar{\theta})$$

To first order, the end results are:

$$[\phi, P_\mu] = i\partial_\mu \phi$$

$$[\phi, \xi^\alpha Q_\alpha] = i\xi^\alpha \left(\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\nu \bar{\theta})_\alpha \partial_\nu \right) \phi$$

$$[\phi, \bar{Q}_{\dot\alpha} \bar{\xi}^{\dot\alpha}] = -i \left(\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} + i(\theta\sigma^\nu)_{\dot\alpha} \partial_\nu \right) \bar{\xi}^{\dot\alpha} \phi$$

<u>Notation:</u> derivatives of Grassmann numbers

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}, \quad \partial^\alpha = \frac{\partial}{\partial\theta_\alpha}, \quad \bar{\partial}_{\dot\alpha} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}, \quad \bar{\partial}^{\dot\alpha} \equiv \frac{\partial}{\partial\bar{\theta}_{\dot\alpha}}$$

$$\partial_\alpha \theta^\beta = \delta_\alpha{}^\beta, \quad \bar{\partial}_{\dot\alpha} \bar{\theta}^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$$

<u>Warning:</u>

$$\partial_\alpha \theta_\beta = \varepsilon_{\beta\alpha} = -\varepsilon_{\alpha\beta}$$

$$\varepsilon^{\alpha\beta} \partial_\beta = -\partial^\alpha$$

Thus, we may represent P^μ, Q_α and $\bar{Q}_{\dot\alpha}$ by differential operators acting on superfields:

$$\hat{P}^\mu = i\partial^\mu$$

$$\hat{Q}_\alpha = i\partial_\alpha - (\sigma^\mu \bar{\theta})_\alpha \partial_\mu$$

$$\hat{\bar{Q}}_{\dot\alpha} = -i\bar{\partial}_{\dot\alpha} + (\theta\sigma^\mu)_{\dot\alpha} \partial_\mu$$

<u>exercise</u>: verify that when acting on a superfield,

$$\{\hat{Q}_\alpha, \hat{\bar{Q}}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} \hat{P}_\mu$$

Expanding the superfield in powers of $\theta, \bar{\theta}$:

Define $\theta\theta \equiv \theta^\alpha \theta_\alpha$

$\bar{\theta}\bar{\theta} \equiv \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$

Then, note that:

$$\theta^\alpha \theta^\beta = -\tfrac{1}{2}\varepsilon^{\alpha\beta}\theta\theta \qquad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \tfrac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}$$

$$\theta_\alpha \theta_\beta = \tfrac{1}{2}\varepsilon_{\alpha\beta}\theta\theta \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}$$

Clearly,

$$\theta^\alpha \theta^\beta \theta^\gamma = 0, \quad \text{etc.}$$

since the θ's anticommute imply that $(\theta^1)^2 = (\theta^2)^2 = 0$.

· Three useful results

(i) $(\theta\sigma^\mu\bar{\theta})\theta_\beta = -\tfrac{1}{2}\theta\theta(\sigma^\mu\bar{\theta})_\beta$

(ii) $(\theta\sigma^\mu\bar{\theta})\bar{\theta}_{\dot{\beta}} = -\tfrac{1}{2}\bar{\theta}\bar{\theta}(\theta\sigma^\mu)_{\dot{\beta}}$

(iii) $(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \tfrac{1}{2}g^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta})$

exercise: prove these results [(iii) follows from the Fierz identities]

Thus, one can expand $\phi(x,\theta,\bar{\theta})$ as a Taylor series in θ and $\bar{\theta}$. The series is finite:

$$\phi(x,\theta,\bar{\theta}) = f(x) + \theta\zeta(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta\, m(x) + \bar{\theta}\bar{\theta}\, n(x)$$
$$+ \theta\sigma^\mu\bar{\theta}\, V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta}\, d(x)$$

where: f, m, n, V_μ, and d are commuting bosonic fields

ζ, χ, λ and ψ are anti-commuting fermionic fields.

How to compute the supersymmetric transformation law

For any superfield Φ,

$$\delta_\xi \Phi = i[\xi Q + \bar{\xi}\bar{Q}, \Phi]$$

But we showed that

$$[\Phi, \xi^\alpha Q_\alpha] = i\xi^\alpha \hat{Q}_\alpha \Phi$$
$$[\Phi, \bar{Q}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}] = i\hat{\bar{Q}}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\Phi$$

where $\hat{Q}_\alpha$, $\hat{\bar{Q}}_{\dot{\alpha}}$ are differential operators.

Thus,

$$\boxed{\delta_\xi \Phi = -i(\xi\hat{Q} + \bar{\xi}\hat{\bar{Q}})\Phi}$$

Insert the expansion

$$\Phi(x,\theta,\bar{\theta}) = f(x) + \theta\zeta(x) + \bar{\theta}\bar{\chi}(x) + \ldots + \theta\theta\bar{\theta}\bar{\theta}d(x)$$

into the above equation and compare like terms in the $\theta, \bar{\theta}$ expansions. This yields the supersymmetric transformation law!

number of degrees of freedom of $\phi(x,\theta,\bar\theta)$

$$\phi(x,\theta,\bar\theta) = f(x) + \theta\zeta(x) + \bar\theta\bar\chi(x) + \theta\theta\, m(x) + \bar\theta\bar\theta\, n(x)$$
$$+ \theta\sigma^\mu\bar\theta\, V_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\psi(x) + \theta\theta\bar\theta\bar\theta\, d(x)$$

boson degrees of freedom

f, m, n, V_μ, d	16	8

fermion degrees of freedom

$\zeta, \chi, \lambda, \psi$	16	8
	↑ if ϕ is complex	↑ if ϕ is real

Note: if ϕ is real, $f = f^*$, $d = d^*$, $n = m^*$, $V_\mu = V_\mu^*$,
$\zeta = \chi$, $\lambda = \psi$.

In either case, this is too many degrees of freedom to describe the supermultiplet of the Wess-Zumino model.

Conclusion: $\phi(x,\theta,\bar\theta)$ describes a reducible representation of supersymmetry. To describe an irreducible representation, one must impose supersymmetric constraints on ϕ.

Remark: the constraint $\phi = \phi^\dagger$ does lead to an irreducible representation. This is the off-shell supermultiplet with $j = \frac{1}{2}$. We will use this later when we study gauge fields.

Fermionic covariant derivatives

If $\phi(x,\theta,\bar\theta)$ is a superfield, it is easy to check that $\partial_\alpha \phi$ is not a superfield, since

$$\partial_\alpha(\delta_\xi \phi) \neq \delta_\xi(\partial_\alpha \phi)$$

[analogy with gauge theory: if ψ is a charged field transforming under a local gauge transformation, $\psi \to e^{i\Lambda(x)}\psi$, then $\partial_\mu\psi$ does not transform as ψ since $\partial_\mu(e^{i\Lambda(x)}\psi) \neq e^{i\Lambda(x)}\partial_\mu\psi$.]

The fermionic covariant derivative D is defined such that

$$D_\alpha(\delta_\xi \phi) = \delta_\xi(D_\alpha \phi)$$

$$\bar{D}_{\dot\alpha}(\delta_\xi \phi) = \delta_\xi(\bar{D}_{\dot\alpha}\phi)$$

Then, $D_\alpha\phi$ and $\bar{D}_{\dot\alpha}\phi$ both transform properly as superfields.

Using $\delta_\xi \phi = i[\xi Q + \bar\xi\bar Q, \phi]$, it follows that

$$D_\alpha(\xi Q + \bar\xi \bar Q) = (\xi Q + \bar\xi\bar Q)D_\alpha \quad , \text{etc.}$$

Since ξ is anticommuting, and $\{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0$ it follows that:

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar Q_{\dot\beta}\} = \{\bar D_{\dot\alpha}, Q_\beta\} = \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0.$$

Using the differential operator representation for $Q, \bar Q$ we can deduce an explicit realization of $D, \bar D$.

$$\boxed{\begin{aligned} D_\alpha &= \partial_\alpha - i(\sigma^\mu \bar\theta)_\alpha \partial_\mu \\ \bar D_{\dot\alpha} &= -\bar\partial_{\dot\alpha} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \end{aligned}}$$

Further manipulation yields:

$$D^\alpha = -\partial^\alpha + i(\bar\theta\bar\sigma^\mu)^\alpha \partial_\mu$$

$$\bar D^{\dot\alpha} = \bar\partial^{\dot\alpha} - i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu$$

where by definition, $D^\alpha \equiv \varepsilon^{\alpha\beta} D_\beta$

$$\bar D^{\dot\alpha} \equiv \varepsilon^{\dot\alpha\dot\beta}\bar D_{\dot\beta}$$

[warning revisited: recall $\varepsilon^{\alpha\beta}\partial_\beta = -\partial^\alpha$, etc.]

Exercise: show that $\{D_\alpha, D_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$

i.e. the same relation satisfied by $\{\hat Q_\alpha, \hat Q_{\dot\beta}\}$.

We are now in the position to impose an interesting constraint on ϕ:

$$\bar D_{\dot\alpha}\phi = 0$$

thereby reducing the number of degrees of freedom. Let's see what happens.

Chiral superfields

$$\bar{D}_{\dot\alpha}\phi = 0$$

$$\left(\frac{-\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\sigma^\mu)_{\dot\alpha}\,\partial_\mu \right)\phi(x,\theta,\bar\theta) = 0$$

solution:

$$\phi(x,\theta,\bar\theta) = \exp\left(-i\theta\sigma^\mu\bar\theta\,\partial_\mu\right)\phi(x,\theta)$$

where $\phi(x,\theta)$ is an arbitrary function of x,θ.

Expand $\phi(x,\theta)$ in a Taylor series in θ.

$$\phi(x,\theta) = A(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x) \qquad \text{[exact!]}$$

(the $\sqrt{2}$ and minus sign are conventional)

Note that:

$$\exp(-i\theta\sigma^\mu\bar\theta\,\partial_\mu) = 1 - i\theta\sigma^\mu\bar\theta\,\partial_\mu - \tfrac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box \qquad \text{[exact!]}$$

[exercise: prove this] $\qquad (\Box \equiv \partial_\mu\partial^\mu)$

Some more manipulation produces:

$$\boxed{\begin{aligned}\phi(x,\theta,\bar\theta) = {} & A(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x) - i\theta\sigma^\mu\bar\theta\,\partial_\mu A(x) \\ & + \frac{i}{\sqrt{2}}\theta\theta\left[\partial_\mu\psi(x)\sigma^\mu\bar\theta\right] - \frac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box A(x)\end{aligned}}$$

The field content: A, ψ, F matches the off-shell fields of the superspin $j=0$ supermultiplet!

<u>exercise:</u> using $\delta_\xi \phi = -i(\xi\hat{Q} + \bar{\xi}\hat{\bar{Q}})\phi$, insert the chiral superfield $\phi(x,\theta,\bar{\theta}) = A + \sqrt{2}\,\theta\psi - \theta\theta F + \ldots$ and derive the supersymmetric transformation laws $\delta_\xi A$, $\delta_\xi \psi$ and $\delta_\xi F$ written down previously.

It is useful to simplify the expression for the chiral superfield by modifying all operators according to:

$$\mathcal{O}_{new} = e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}\,\mathcal{O}\,e^{-i\theta\sigma^\mu\bar{\theta}\partial_\mu}$$

In the new "chiral representation",

$$\hat{Q}_\alpha = i\partial_\alpha$$
$$\hat{\bar{Q}}_{\dot{\alpha}} = -i\bar{\partial}_{\dot{\alpha}} + 2(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu$$
$$D_\alpha = \partial_\alpha - 2i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu$$
$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}}$$

Now, the condition $\bar{D}_{\dot{\alpha}}\phi = 0$ simply means that ϕ is independent of $\bar{\theta}$. Let us call the chiral superfield

$$\phi_1(x,\theta) = A(x) + \sqrt{2}\,\theta\psi(x) - \theta\theta F(x)$$

in the chiral representation. Clearly,

$$\phi(x,\theta,\bar{\theta}) = \exp(-i\theta\sigma^\mu\bar{\theta}\partial_\mu)\,\phi_1(x,\theta)$$
$$= \phi_1(x - i\theta\sigma^\mu\bar{\theta},\ \theta)$$

Working in the chiral representation can simplify many calculations.

Anti-chiral superfields

$$D_\alpha \bar{\Phi} = 0$$

$$\bar{\Phi}(x,\theta,\bar{\theta}) = \exp(i\theta\sigma^\mu\bar{\theta}\partial_\mu)\bar{\phi}(x,\bar{\theta})$$

$$= A^*(x) + \sqrt{2}\,\bar{\theta}\bar{\Psi}(x) - \bar{\theta}\bar{\theta}F^*(x) + \ldots$$

One can define the anti-chiral representation where

$$\hat{Q}_\alpha = i\partial_\alpha - 2(\sigma^\mu\bar{\theta})_\alpha\partial_\mu$$

$$\hat{\bar{Q}}_{\dot{\alpha}} = -i\bar{\partial}_{\dot{\alpha}}$$

$$D_\alpha = \partial_\alpha$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + 2i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu$$

and

$$\phi_2(x,\bar{\theta}) = A^*(x) + \sqrt{2}\,\bar{\theta}\bar{\Psi}(x) - \bar{\theta}\bar{\theta}F^*(x)$$

Then,

$$\phi(x,\theta,\bar{\theta}) = \exp(i\theta\sigma^\mu\bar{\theta}\partial_\mu)\phi_2(x,\bar{\theta})$$

$$= \phi_2(x + i\theta\sigma^\mu\bar{\theta}, \bar{\theta})$$

Some jargon:

The F-component of a chiral superfield is the coefficient of the $\theta\theta$ term (with an extra minus sign).

Sometimes I will write:

$$[\Phi]_{\theta\theta} = -F .$$

Theorem: If Φ is a chiral superfield then so is Φ^n.

But $\Phi^n \bar{\Phi}^m$ is not a chiral superfield (n, m are positive integers).

proof: $\bar{D}_{\dot{\alpha}} \Phi^n = n \Phi^{n-1} \bar{D}_{\dot{\alpha}} \Phi = 0 .$

Theorem: F-terms transform as total derivatives.

recall: $\delta_\xi F = -i\sqrt{2}\, \partial_\mu \psi^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}}$

but now we identify

$$[\Phi]_\theta = \sqrt{2}\, \psi(x)$$
$$[\Phi]_{\theta\theta} = -F$$

Consequently,

$$\sum_{n\geq 1} [a_n \Phi^n]_{\theta\theta} + h.c.$$

is a Lorentz scalar that transforms as a total divergence, and is thus a possible term in a supersymmetric Lagrangian.

exercise: multiply together two chiral multiplets. Do the computation in the chiral representation.

$$\phi_a(x) = A_a(x) + \sqrt{2}\,\theta\psi_a(x) - \theta\theta F_a(x)$$
$$\phi_b(x) = A_b(x) + \sqrt{2}\,\theta\psi_b(x) - \theta\theta F_b(x)$$

Show that:

$$[\phi_a\phi_b]_{\theta\theta} = -[A_a(x)F_b(x) + A_b(x)F_a(x) + \psi_a(x)\psi_b(x)]$$

Similarly,

$$[\phi_a\phi_b\phi_c]_{\theta\theta} = -[A_aA_bF_c + A_aA_cF_b + A_bA_cF_a + A_a\psi_b\psi_c + A_b\psi_a\psi_c + A_c\psi_a\psi_b]$$

The kinetic superfield

$$T\phi \equiv \tfrac{1}{4}\bar{D}^2\bar{\phi}$$

where $\bar{D}^2 \equiv \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}$, and $\bar{\phi}$ is anti-chiral.

Note that $\bar{D}_{\dot\alpha}(T\phi) = 0$ so that $T\phi$ is chiral.

An explicit computation gives [in the chiral representation]:

If $\phi = A + \sqrt{2}\,\theta\psi - \theta\theta F$, then

$$T\phi = F^* + i\sqrt{2}\,\theta\sigma^\mu\partial_\mu\bar{\psi} + \theta\theta\,\Box A^*$$

Let us compute:

$$[\phi\, T\phi]_F = -[\phi\, T\phi]_{\theta\theta}$$

$$[\phi T \phi]_F = -A \Box A^* + F^* F + i \psi \sigma^\mu \partial_\mu \bar{\psi}$$

$$= (\partial_\mu A)(\partial^\mu A^*) + F^* F + i \bar{\psi} \sigma^\mu \partial_\mu \psi$$

$$+ \text{ total divergence}$$

which we recognize as the kinetic energy of the Wess-Zumino model.

<u>Theorem</u>: For any chiral superfield,

$$[\phi]_F = \frac{1}{4} D^2 \phi \Big|_{\theta=\bar{\theta}=0} = -\frac{1}{4} \partial^\alpha \partial_\alpha \phi \Big|_{\theta=\bar{\theta}=0}$$

since $D_\alpha = \partial_\alpha + \theta$-dependent terms.

Given any holomorphic function of the chiral superfield, $W(\phi)$, we can compute:

$$[W(\phi)]_F = -\frac{1}{4} \partial^\alpha \partial_\alpha W \Big|_{\theta=\bar{\theta}=0}$$

$$= -\frac{1}{4} \partial^\alpha \frac{\partial W}{\partial \phi} \partial_\alpha \phi \Big|_{\theta=\bar{\theta}=0} \qquad \text{by the chain rule}$$

$$= -\frac{1}{4} \left(\frac{\partial^2 W}{\partial \phi^2} \partial^\alpha \phi \partial_\alpha \phi \right) + \frac{\partial W}{\partial \phi} \frac{1}{4} D^2 \phi \Big|_{\theta=\bar{\theta}=0}$$

Noting that:

$$\partial^\alpha \phi \partial_\alpha \phi \Big|_{\theta=\bar{\theta}=0} = -\varepsilon^{\alpha\beta} \partial_\beta \phi \partial_\alpha \phi \Big|_{\theta=\bar{\theta}=0} = -2\psi\psi$$

Conclusion:

$$[W(\Phi)]_F = \frac{1}{2}\frac{d^2W}{dA^2}\psi\psi + \frac{dW}{dA}F$$

where $\frac{dW}{dA}$ means: replace Φ with A in $W(\Phi)$ and then compute the derivative.

The supersymmetric Lagrangian

$$\mathcal{L} = [\Phi^\dagger \Phi]_F - \left([W(\Phi)]_F + h.c.\right)$$

$$= (\partial_\mu A)^*(\partial^\mu A) + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^*F$$

$$-F\frac{dW}{dA} - F^*\left(\frac{dW}{dA}\right)^* - \frac{1}{2}\left[\frac{d^2W}{dA^2}\psi\psi + \left(\frac{d^2W}{dA^2}\right)^*\bar{\psi}\bar{\psi}\right]$$

which recovers our previous result.

Proof that $\delta_\xi\mathcal{L} = \partial_\mu\tilde{K}^\mu$.

immediate! since $\mathcal{L}$ is the sum of F (and F^*) terms.

Some supersymmetric jargon: $W(\Phi)$ is the superpotential.

It is a holomorphic function of chiral superfields.

R-invariance

Recall $[R, Q_\alpha] = -Q_\alpha$

$[R, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}$

As a differential operator acting on superspace,

$$[\phi, R] = (\theta^\alpha \partial_\alpha - \bar\theta_{\dot\alpha}\bar\partial^{\dot\alpha} - 2n)\phi$$

$2n$ = chiral weight of the superfield ϕ. $\quad n \in \mathbb{R}$.

i.e. $\hat{R} = \theta\partial - \bar\theta\bar\partial - 2n$

Under an R-transformation,

$$\delta_a \phi = ia[R, \phi] = -ia(\theta\partial - \bar\theta\bar\partial - 2n)\phi$$

Acting on a chiral superfield or anti-chiral superfield

$$\hat{R}\phi(x,\theta) = e^{2ina}\phi(x, e^{-ia}\theta)$$
$$\hat{R}\bar\Phi(x,\bar\theta) = e^{-2ina}\bar\Phi(x, e^{ia}\bar\theta)$$

Infinitesimally,

$$\delta_a A = 2ina\, A$$
$$\delta_a \psi = (2n-1)ia\, \psi$$
$$\delta_a F = (2n-2)ia\, F$$

$2n$ = R-charge of the superfield ϕ.

<u>Question</u>: is $\delta_a \mathcal{L} = 0$?

<u>exercise</u>: prove that $[\Phi T \Phi]_F$ is R-invariant.

Note: the fully exponentiated form of the R-transformation is:

$$A \to e^{2ina} A$$
$$\psi \to e^{2i(n-\frac{1}{2})a} \psi$$
$$F \to e^{2i(n-1)a} F.$$

Note: $[W]_F$ is R-invariant only if W has R-charge, $2n$, equal to 2.

proof: this follows immediately from the transformation of F.

example: the Wess-Zumino model with

$$W(\phi) = \frac{1}{2} m \phi^2 + \frac{1}{3} g \phi^3$$

If $m=0$, $\mathcal{L}$ is R-invariant with the choice $n=\frac{1}{3}$.
If $g=0$, $\mathcal{L}$ is R-invariant with the choice $n=\frac{1}{2}$.

If both $m \neq 0$ and $g \neq 0$, then $\mathcal{L}$ is not R-invariant.

Finally, it should be noted that for a real superfield (not chiral or anti-chiral), the only possible choice for the R-charge is $n=0$.

$$\hat{R}\phi(x,\theta,\bar{\theta}) = \phi(x, e^{-ia}\theta, e^{ia}\bar{\theta})$$

if $\phi^\dagger = \phi$.

Grassmann integration

The rules:

$$\int d\theta^\alpha = \int d\bar{\theta}^{\dot{\alpha}} = 0$$

$$\int d\theta^\alpha \, \theta^\alpha = \int d\bar{\theta}^{\dot{\alpha}} \, \bar{\theta}^{\dot{\alpha}} = 1 \qquad \text{no sum over } \alpha, \dot{\alpha}.$$

definitions:

$$d^2\theta = -\tfrac{1}{4} d\theta^\alpha d\theta_\alpha$$

$$d^2\bar{\theta} = -\tfrac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}^{\dot{\alpha}}$$

$$d^4\theta = d^2\theta \, d^2\bar{\theta}$$

Then,

$$\int d^2\theta \; \theta\theta = \int d^2\bar{\theta} \; \bar{\theta}\bar{\theta} = \int d^4\theta \; \theta\theta\bar{\theta}\bar{\theta} = 1$$

and all other integrals equal zero.

It follows that for a chiral superfield,

$$\int d^2\theta \, \phi(x,\theta) = [\phi]_{\theta\theta} = -\tfrac{1}{4} D^2\phi \Big|_{\theta=\bar{\theta}=0}$$

while for an unconstrained superfield

$$\int d^4\theta \, \phi(x,\theta,\bar{\theta}) = [\phi]_{\theta\theta\bar{\theta}\bar{\theta}}$$

We can write a supersymmetric action as an integral over superspace. Simply note that:

$$[\bar{\Phi}\Phi]_{\theta\theta\bar{\theta}\bar{\theta}} = -[\phi T \phi]_{\theta\theta}$$

The Wess-Zumino action $S = \int d^4x \mathcal{L}$ can be written as:

$$S = \int d^4x\, d^4\theta\, \bar{\Phi}\phi + \int d^4x\, d^2\theta\, W(\phi) + \int d^4x\, d^2\bar{\theta}\, W(\bar{\Phi}).$$

which is manifestly supersymmetric.

Note: We have already noted that F-terms transform as total divergences. For a general superfield

$$\phi = \ldots + \theta\theta\bar{\theta}\bar{\theta} D$$

we can work out $\delta_\xi D$ and show that

$$[\phi]_D = [\phi]_{\theta\theta\bar{\theta}\bar{\theta}}$$

transforms as a total divergence. Thus, D-terms also make for suitable terms of a supersymmetric Lagrangian.

Example: $[\bar{\Phi}\Phi]_D = (\partial_\mu A)(\partial^\mu A^*) + i\bar{\Psi}\bar{\sigma}^\mu\partial_\mu\Psi + F^*F$
+ total divergence

as expected.

The effective supersymmetric action

Recall the effective action of quantum field theory – this is a derivative expansion and thus serves as a good approximation at low energies. It incorporates radiative corrections (via the loop expansions as an expansion in powers of $\hbar$) in each term:

$$\Gamma[\phi] = \int d^4x \left[\frac{1}{2} Z_{eff}(\phi)\, \partial_\mu \phi \partial^\mu \phi - V_{eff}(\phi) + \text{higher derivative terms} \right]$$

The corresponding expansion in supersymmetric field theory (keeping only terms up to two derivatives) takes the form:

$$\Gamma[\Phi] = \int d^4x\, d^4\theta\, K(\bar{\Phi}, \Phi)$$

$$+ \int d^4x\, d^2\theta\, W_{eff}(\Phi) + \int d^4x\, d^2\bar{\theta}\, W_{eff}(\bar{\Phi})$$

where the effective superpotential $W_{eff}(\Phi)$ is a holomorphic function of Φ and $K(\bar{\Phi}, \Phi)$ is called the Kähler potential.

(i) $K(\bar{\Phi}, \Phi) = \bar{\Phi}\Phi + o(\hbar)$

(ii) $W_{eff}(\Phi) = W(\Phi)$

Statement (ii) is the famous "no-renormalization" theorem of $N=1$ Supersymmetry.

The Kähler potential

$$[K(\Phi,\bar\Phi)]_D = \frac{1}{16}\bar D^2 D^2 K(\Phi,\bar\Phi)\Big|_{\theta=\bar\theta=0}$$

Applying the chain rule (as before — see the computation of $[W(\Phi)]_F$), and after much algebra,

$$[K(\Phi,\bar\Phi)]_D = \frac{\partial^2 K}{\partial A\,\partial A^*}\left[(\partial_\mu A^*)(\partial^\mu A) + F^*F + \frac{i}{2}\bar\psi\bar\sigma^\mu \overleftrightarrow{\partial}_\mu \psi\right]$$
$$+\frac{1}{2}\frac{\partial^3 K}{\partial A\,\partial A^{*2}}\left[F\bar\psi\bar\psi - i\bar\psi\bar\sigma^\mu\psi\,\partial_\mu A^*\right]$$
$$+\frac{1}{2}\frac{\partial^3 K}{\partial A^2\partial A^*}\left[F^*\psi\psi + i\bar\psi\bar\sigma^\mu\psi\,\partial_\mu A\right]$$
$$+\frac{1}{4}\frac{\partial^4 K}{\partial A^2\partial A^{*2}}(\psi\psi)(\bar\psi\bar\psi) + \text{total divergence}$$

If one includes the superpotential term in the effective action, one can eliminate the auxiliary field F as usual:

$$F^* = \left(\frac{\partial^2 K}{\partial A\,\partial A^*}\right)^{-1}\left[\frac{dW}{dA} - \frac{1}{2}\frac{\partial^3 K}{\partial A\,\partial A^{*2}}\bar\psi\bar\psi\right]$$

to get the Lagrangian in terms of physical fields.

The Kähler potential arises in:

- low-energy effective field theories (that include higher dimensional operators)
- supersymmetric σ-models
- theories of supergravity.

Non-renormalization theorem

The superpotential is not renormalized to all orders in perturbation theory. Only the kinetic energy term is renormalized - this is wave function renormalization.

Hence, no quadratic divergences in the parameters of the superpotential. Mass renormalization is generated simply as a consequence of wave function renormalization, namely

$$m_B^2 \phi_B^2 = m_R^2 \phi_R^2$$

and $\phi_B = Z_\phi^{1/2} \phi_R$, which implies that

$$m_R^2 = Z_\phi m_B^2 .$$

Wave-function renormalization is logarithmic in the cutoff.

The original proof of this theorem demonstrated that all radiative corrections could be written in the form of an integral over $d^4\theta$. Thus, terms of the form

$$\int d^4x \, d^2\theta \, W(\phi)$$

are not affected.