Introduction to Supersymmetry

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Abstract

These are the (mostly handwritten) notes of four introductory lectures to supersymmetry. Lecture 1 provides a motivation for low-energy supersymmetry and introduces two-component technology for spin-1/2 fermions in quantum field theory. Lecture 2 discusses the supersymmetric extension of the Poincare algebra and introduces the concepts of superspace and chiral superfields. Lecture 3 treats supersymmetric gauge theories and supersymmetry breaking. Finally, Lecture 4 introduces the minimal supersymmetric extension of the Standard Model (MSSM) and describes how this model must be constrained in order to be phenomenologically viable. If there is time, some topics beyond the MSSM (non-minimal extensions, R-parity violation and grand unified models) will be discussed.

Lectures given at the Zuoz 2004 Summer School
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Outline of Topics

1. Motivation for "Low-Energy" Supersymmetry
   - Hierarchy, naturalness and all that
   - Beyond the Standard Model: expectations for new physics

2. Spin-1/2 fermions in quantum field theory
   - Two-component technology
   - Feynman rules for Majorana fermions

3. Supersymmetry—first steps
   - Extending Poincaré invariance
   - Simple supersymmetric models of fermions and their scalar superpartners

4. Superspace and chiral superfields
   - How to construct a supersymmetric action
   - Non-renormalization theorems

5. Supersymmetric gauge field theories
    - The marriage of supersymmetry and gauge theories
    - Supersymmetric QED
    - Supersymmetric Yang Mills theories

6. Supersymmetry breaking
    - spontaneous supersymmetry breaking
    - generic soft-supersymmetry-breaking terms

7. Supersymmetric extension of the Standard Model
    - The minimal supersymmetric extension (MSSM)
    - A tour of the supersymmetric spectrum
    - The MSSM Higgs sector

8. Constraining the low-energy supersymmetric model
    - Counting the MSSM parameters
    - Phenomenological disasters and how to avoid them
    - A brief look at fundamental theories of supersymmetry breaking

A Supersymmetric Bibliography

1. Recommended texts:

Supersymmetric Gauge Field Theory and String Theory, by D. Bailin and A. Love

Supersymmetry and Supergravity, by Julius Wess and Jonathan Bagger

The Quantum Theory of Fields, Volume III: Supersymmetry, by Steven Weinberg

Supersymmetry, Superfields and Supergravity: An Introduction, by Prem P. Srivastava

Supersymmetry: An Introduction with Conceptual and Calculational Details, by H.J.W. Müller-Kirsten and A. Wiedemann

Introduction to Supersymmetry and Supergravity, by S.P. Misra

Supersymmetry: Basics and Concepts, by S.K. Soni and Simmi Singh

2. More formal treatments:

Introduction to Supersymmetry, by Peter G.O. Freund

Introduction to Supersymmetry and Supergravity, 2nd edition, by Peter West

Ideas and Methods of Supersymmetry and Supergravity, by Ioseph L. Buchbinder and Sergei M. Kuzenko

Superspace or One Thousand and One Lessons in Supersymmetry, by S. James Gates, Jr., Marcus T. Grisaru, Martin Rocek and Warren Siegel [arXiv: hep-th/0108200].

3. Review Articles from *Physics Reports*:

1. *Supersymmetry*, Pierre Fayet and Sergio Ferrara, *Physics Reports* **32** (1977) 249.

2. *Supersymmetry, Supergravity and Particle Physics*, Hans Peter Nilles, *Physics Reports* **110** (1984) 1.

3. *The Search for Supersymmetry: Probing Physics beyond the Standard Model*, Howard E. Haber and Gordon L. Kane, *Physics Reports* **117** (1985) 75.

4. *Introducing Supersymmetry*, Martin F. Sohnius, *Physics Reports* **128** (1985) 39.

5. *Supersymmetry and Supergravity: A Reprint Volume of Physics Reports*, edited by Maurice Jacob.

4. Other Review Articles:

1. *Introductory Low-Energy Supersymmetry*, by Howard E. Haber
in *Recent Directions in Particle Theory*, Proceedings of the 1992 Theoretical Advanced Study Institute, edited by Jeffrey Harvey and Joseph Polchinski
[arXiv: hep-ph/9306207].

2. *Introductory lectures on low-energy supersymmetry*, by Pierre Ramond
in *CP Violation and the limits of the Standard Model*, Proceedings of the 1994 Theoretical Advanced Study Institute, edited by John F. Donoghue [arXiv:hep-th/9412234].

3. *Introduction to Supersymmetry*, by Joseph Lykken
in *Fields, Strings and Duality*, Proceedings of the 1996 Theoretical Advanced Study Institute, edited by Costas Efthimiou and Brian Greene [arXiv: hep-th/9612114].

4. *A Supersymmetry Primer*, by Stephen P. Martin, arXiv: hep-ph/9709356.

5. *Introduction to supersymmetric gauge theories*, by Olivier Piguet
Lectures given at the 1st School on Field Theory and Gravitation, Vitoria, Brazil, 15–19 April 1997, arXiv:hep-th/9710095.

6. *Supersymmetry: Structure and phenomena: Extensions of the standard model*, by N. Polonsky, *Lecture Notes in Physics* **M68** (2001) [arXiv:hep-ph/0108236].

7. *Introduction to supersymmetry*, by Adel Bilal
lectures given at the Summer School GIF 2000, arXiv:hep-th/0101055.

8. *BUSSTEPP lectures on supersymmetry*, by José Miguel Figueroa-O'Farrill
arXiv:hep-th/0109172.

9. *An Unorthodox Introduction to Supersymmetric Gauge Theory*, by Matthew Strassler
in *Strings, Branes and Extra Dimensions*, Proceedings of the 2001 Theoretical Advanced Study Institute, edited by Steve S. Gubser and Joseph D. Lykken [arXiv:hep-th/0309149].

10. *TASI-2002 lectures: Non-perturbative supersymmetry*, by John Terning
in *Particle Physics and Cosmology: The Quest for Physics Beyond the Standard Model(s)*, Proceedings of the 2002 Theoretical Advanced Study Institute, edited by Howard E. Haber and Ann Nelson [arXiv:hep-th/0306119].

I. MOTIVATION FOR LOW-ENERGY SUPERSYMMETRY

The Standard Model of particle physics is a superb description of fundamental particles and their interactions ... with two notable footnotes:

(i) neutrinos are not exactly massless

suggestive, perhaps, of a new high-energy scale much larger than 1 TeV. It is fairly easy to extend the Standard Model to account for massive neutrinos, and we will return to this point later.

(ii) the Higgs boson has not yet been discovered

although precision electroweak data is consistent with $m_h \lesssim 250$ GeV.

$$\boxed{\text{Nevertheless, the Standard Model (including the Higgs boson) cannot be considered to be a truly fundamental theory of particle physics, valid to arbitrarily high energies.}}$$

At best, the Standard Model is an effective theory, which is valid up to some energy scale Λ

At energy scales above Λ, new physics enters and the Standard Model is no longer adequate for describing fundamental physics.

At energies below Λ, the Standard Model is a very good approximation for the theory of fundamental particles and their interactions. There will be deviations, but they are suppressed by at least a factor of E/Λ at an energy scale E.

These deviations should be small at "low energies" $E \ll \Lambda$. In this language, the energy scale that characterizes electroweak physics, $v \simeq 246$ GeV, is low energy.

<u>example</u>: neutrino masses generated by the see-saw mechanism are of $O\left(\frac{v^2}{\Lambda}\right)$ where $\Lambda \sim 10^{15}$ GeV

What's missing?

So far, gravity is not yet included. Quantum gravitational effects are relevant only at a very high energy scale, called the Planck scale

$$M_{\rm PL} = (c\hbar/G_N)^{1/2} \simeq 10^{19}\ {\rm GeV}\,,$$

which arises as follows. The gravitational potential energy of a particle of mass M, $G_N M^2/r$ (where G_N is Newton's gravitational constant), evaluated at its Compton wavelength, $r = \hbar/Mc$, is of order the rest mass, Mc^2, when

$$G_N M^2 \left(\frac{Mc}{\hbar}\right) \sim Mc^2\,,$$

which implies that $M^2 \sim c\hbar/G_N$. When this happens, the gravitational energy is large enough to induce pair production, which means that quantum gravitational effects can no longer be neglected. Thus, the Planck scale, $M_{\rm PL} = (c\hbar/G_N)^{1/2}$, represents the energy scale at which gravity and all other forces of elementary particles must be incorporated into the same theory.

Where does the Standard Model Break Down?

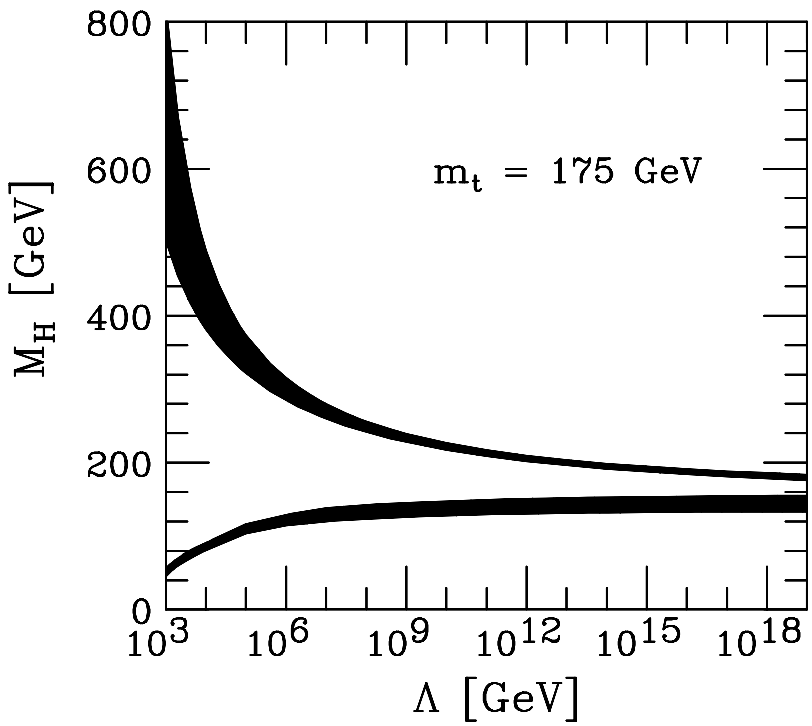
The Standard Model (SM) describes quite accurately physics near the electroweak symmetry breaking scale [$v = 246$ GeV]. But, the SM is only a "low-energy" approximation to a more fundamental theory.

- The Standard Model cannot be valid at energies above the Planck scale, $M_{\rm PL} \equiv (c\hbar/G_N)^{1/2} \simeq 10^{19}$ GeV, where gravity can no longer be ignored.

- Neutrinos are exactly massless in the Standard Model. But, recent experimental observations of neutrino mixing imply that neutrinos have very small masses ($m_\nu/m_e \lesssim 10^{-7}$). Neutrino masses can be incorporated in a theory whose fundamental scale is $M \gg v$. Neutrino masses of order v^2/M are generated, which suggest that $M \sim 10^{15}$ GeV.

- When radiative corrections are evaluated, one finds:
  - The Higgs potential is unstable at large values of the Higgs field ($|\Phi| > \Lambda$) if the Higgs mass is too small.
  - The value of the Higgs self-coupling runs off to infinity at an energy scale above Λ if the Higgs mass is too large.

This is evidence that the Standard Model must break down at energies above Λ.



The present-day theoretical uncertainties on the lower [Altarelli and Isidori; Casas, Espinosa and Quirós] and upper [Hambye and Riesselmann] Higgs mass bounds as a function of energy scale Λ at which the Standard Model breaks down, assuming $m_t = 175$ GeV and $\alpha_s(m_Z) = 0.118$. The shaded areas above reflect the theoretical uncertainties in the calculations of the Higgs mass bounds.

Significance of the TeV Scale

In 1939, Weisskopf computed the self-energy of a Dirac fermion and compared it to that of an elementary scalar. The fermion self-energy diverged logarithmically, while the scalar self-energy diverged quadratically. If the infinities are cut-off at a scale Λ, then Weisskopf argued that for the particle mass to be of order the self-energy,

- For the e^-, $\Lambda \sim m \exp(\alpha^{-1}) \gg M_{\rm PL}$ $[\alpha \equiv e^2/4\pi \simeq 1/137]$;
- For an elementary boson, $\Lambda \sim m/g$, where g is the coupling of the boson to gauge fields.

In modern times, this is called the hierarchy and naturalness problem. Namely, how can one understand the large hierarchy of energy scales from v to $M_{\rm PL}$ in the context of the SM? If the SM is superseded by a more fundamental theory at an energy scale Λ, one expects scalar squared-masses to exhibit at one-loop order quadratic sensitivity to Λ, (in contrast to the logarithmic sensitivity of the fermions). That is, the *natural* value for the scalar squared-mass is roughly $(g^2/16\pi^2)\Lambda^2$. Thus,

$$\boxed{\Lambda \simeq 4\pi m_h/g \sim \mathcal{O}(1~\text{TeV}).}$$

On the Self-Energy and the Electromagnetic Field of the Electron

V. F. WEISSKOPF
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(Received April 12, 1939)

The charge distribution, the electromagnetic field and the self-energy of an electron are investigated. It is found that, as a result of Dirac's positron theory, the charge and the magnetic dipole of the electron are extended over a finite region; the contributions of the spin and of the fluctuations of the radiation field to the self-energy are analyzed, and the reasons that the self-energy is only logarithmically infinite in positron theory are given. It is proved that the latter result holds to every approximation in an expansion of the self-energy in powers of e^2/hc. The self-energy of charged particles obeying Bose statistics is found to be quadratically divergent. Some evidence is given that the "critical length" of positron theory is as small as $h/(mc)\cdot\exp(-hc/e^2)$.

I. Introduction and Discussions of Results

The self-energy of the electron is its total energy in free space when isolated from other particles or light quanta. It is given by the expression

$$W = T + (1/8\pi)\int (H^2+E^2)d\mathbf{r}. \tag{1}$$

Here T is the kinetic energy of the electron; H and E are the magnetic and electric field strengths. In classical electrodynamics the self-energy of an electron of radius a at rest and without spin is given by $W \sim mc^2 + e^2/a$ and consists solely of the energy of the rest mass and of its electrostatic field. This expression diverges linearly for an infinitely small radius. If the electron is in motion, other terms appear representing the energy produced by the magnetic field of the moving electron. These terms, of course, can be obtained by a Lorentz transformation of the former expression.

The quantum theory of the electron has put the problem of the self-energy in a critical state. There are three reasons for this:

(a) Quantum kinematics shows that the radius of the electron must be assumed to be zero. It is easily proved that the product of the charge densities at two different points, $\rho(\mathbf{r}-\xi/2)\times\rho(\mathbf{r}+\xi/2)$, is a delta-function $e^2\delta(\xi)$. In other words: if one electron alone is present, the probability of finding a charge density simultaneously at two different points is zero for every finite distance between the points. Thus the energy of the electrostatic field is infinite as

$$W_{\mathrm{st}} = \lim_{(a=0)} e^2/a.$$

(b) The quantum theory of the relativistic electron attributes a magnetic moment to the electron, so that an electron at rest is surrounded by a magnetic field. The energy

$$U_{\mathrm{mag}} = (1/8\pi)\int H^2 d\mathbf{r}$$

of this field is computed in Section III and the result is

$$U_{\mathrm{mag}} = e^2h^2/(6\pi m^2c^2a^3).$$

This corresponds to the field energy of a magnetic dipole of the moment $eh/2mc$ which is spread over a volume of the dimensions a. The spin, however, does not only produce a magnetic field, it also gives rise to an alternating electric field. The closer analysis of the Dirac wave equation has shown[1] that the magnetic moment of the spin is produced by an irregular circular fluctuation movement (Zitterbewegung) of the electron which is superimposed to the translatory motion. The instantaneous value of the velocity is always found to be c. It must be expected that this motion will also create an alternating electric field. The existence of this field is demonstrated in Section III by the computation of the expression

$$U_{\mathrm{el}} = (1/8\pi)\int E_s^2 d\mathbf{r}.$$

There E_s is the solenoidal part (div. $E_s=0$) of the electric field strength created by the electron. The fact that the above expression does not vanish for an electron at rest proves the existence

[1] E. Schroedinger, Berl. Ber. 1930, 418 (1930).

zero in the one-electron theory, is negative and quadratically divergent in the positron theory. This is because of the negative contribution of the magnetic field and the interference effect of the electric field of the vacuum electrons.

(c) The energy W_{fluct} of forced vibrations under the influence of the zero-point fluctuations of the radiation field. The energies (b) and (c) compensate each other to a logarithmic term.

It is interesting to apply similar considerations to the scalar theory of particles obeying the Bose statistics, as has been developed by Pauli and the author.[7] Here the probability of finding two equal particles closer than their wave-lengths is *larger* than at longer distances. The effect on the self-energy is therefore just the opposite. The influence of the particle on the vacuum causes a higher singularity in the charge distribution instead of the hole which balanced the original charge in the previous considerations. It is shown in Section V that this gives rise to a quadratically divergent energy of the Coulomb field of the particle. Thus the situation here is even worse than in the classical theory. The spin term obviously does not appear and the energy W_{fluct} is exactly equal to its value for a Fermi particle.

A few remarks might be added about the possible significance of the logarithmic divergence of the self-energy for the theory of the electron. It is proved in Section VI that every term in the expansion of the self-energy in powers of e^2/hc

$$W=\sum_n W^{(n)} \tag{3}$$

diverges logarithmically with infinitely small electron radius and is approximately given by

$$W^{(n)}\sim z_n mc^2(e^2/hc)^n[\lg\,(h/mca)]^t, \quad t\leqq n.$$

Here the z_n are dimensionless constants which cannot easily be computed. It is therefore not sure, whether the series (3) converges even for finite a, but it is highly probable that it converges if $\delta=e^2/(hc)\cdot\lg\,(h/mca)<1$. One then would get $W=mc^2O(\delta)$ where $O(\delta)=1$ for a value of $\delta<1$. We then can define an electron radius in the same way as the classical radius e^2/mc^2 is defined, by putting the self-energy equal to mc^2. One obtains then roughly a value $a\sim h/(mc)\cdot\exp\,(-hc/e^2)$ which is about 10^{-58} times smaller than the classical electron radius. The "critical length" of the positron theory is thus infinitely smaller than usually assumed.

The situation is, however, entirely different for a particle with Bose statistics. Even the Coulombian part of the self-energy diverges to a first approximation as $W_{\text{st}}\sim e^2h/(mca^2)$ and requires a much larger critical length that is $a=(hc/e^2)^{-\frac{1}{2}}\cdot h/(mc)$, to keep it of the order of magnitude of mc^2. This may indicate that a theory of particles obeying Bose statistics must involve new features at this critical length, or at energies corresponding to this length; whereas a theory of particles obeying the exclusion principle is probably consistent down to much smaller lengths or up to much higher energies.

[7] W. Pauli and V. Weisskopf, Helv. Phys. Acta **7**, 709 (1934).

II. The Charge Distribution of the Electron

The charge distribution in the neighborhood of an electron can be determined from the expression

$$G(\xi)=\int\rho(\mathbf{r}-\xi/2)\rho(\mathbf{r}+\xi/2)d\mathbf{r}; \tag{4}$$

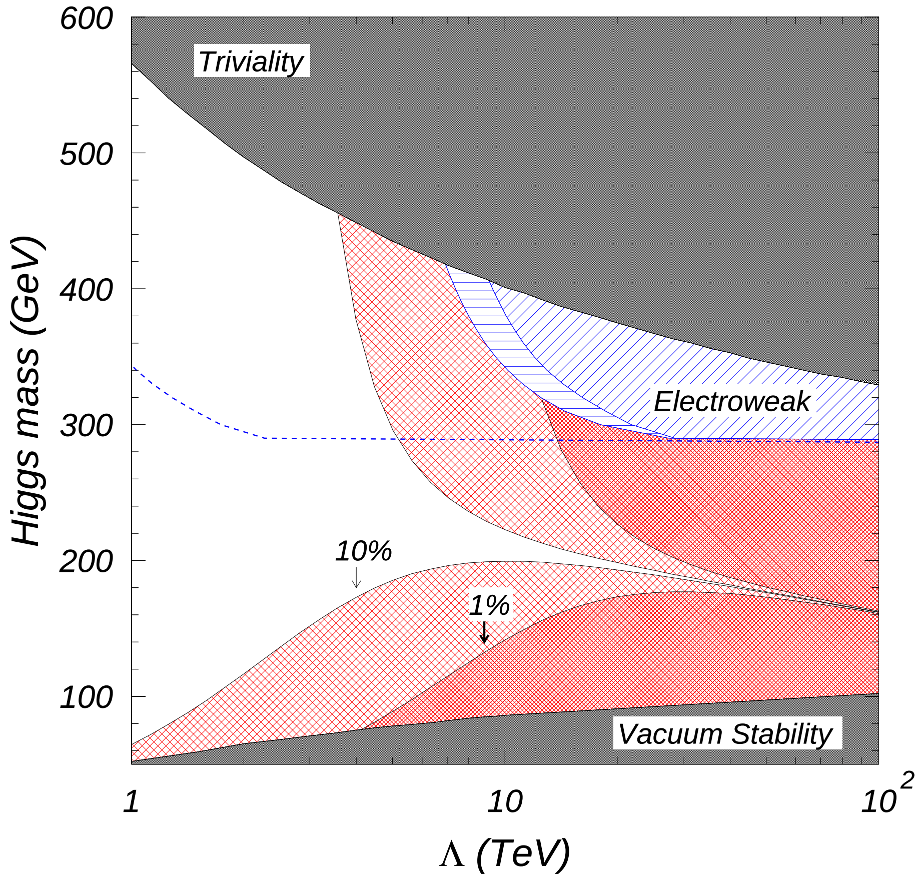
here $\rho(\mathbf{r})$ is the charge density at the point $\mathbf{r}$. $G(\xi)$ is the probability of finding charge simultaneously at two points in a distance ξ. If applied to a situation in which one electron alone is present, direct information can be drawn from this expression concerning the charge distribution in the electron itself. The charge density is given by

$$\rho(\mathbf{r})=e\{\psi^*(\mathbf{r})\psi(\mathbf{r})\}-\sigma, \tag{5}$$

where $\psi(\mathbf{r})$, the wave function, is a spinor with four components ψ_μ, $\mu=1, 2, 3, 4$. We write

$$\{\psi^*\psi\}=\sum_{\mu=1}^{4}\psi_\mu{}^*\psi_\mu$$

for the scalar product of two spinors. σ is the charge density of the unperturbed electrons in the negative energy states which is to be subtracted in the positron theory. In the one-electron theory σ is zero. The wave function ψ can be expanded in wave functions φ_q of the



Following Kolda and Murayama [*JHEP* **0007** (2000) 035], a reconsideration of the Λ *vs.* Higgs mass plot with a focus on $\Lambda < 100$ TeV. Precision electroweak measurements restrict the parameter space to lie below the dashed line, based on a 95% CL fit that includes the possible existence of higher dimensional operators suppressed by v^2/Λ^2. The unshaded area has less than one part in ten fine-tuning.

If we demand that the value of m_h is natural, *i.e.*, without substantial fine-tuning, then Λ cannot be significantly larger than 1 TeV. Since Λ represents the scale at which new physics beyond the SM must enter, which will provide a natural explanation for the value of m_h (or more generally, the scale of electroweak symmetry breaking), we should ask:

What new physics is lurking at the TeV scale?

Can quadratic sensitivity to Λ be avoided in a theory with elementary scalars?

A lesson from history

The electron self-energy in classical electromagnetism goes like e^2/a $(a \to 0)$, *i.e.*, it is linearly divergent. In quantum theory, fluctuations of the electromagnetic fields (in the "single electron theory") generate a quadratic divergence. If these divergences are not canceled, one would expect that QED should break down at an energy of order m_e/e far below the Planck scale (a severe hierarchy problem).

The linear and quadratic divergences will cancel exactly if one makes a bold hypothesis: the existence of the positron (with a mass equal to that of the electron but of opposite charge).

Weisskopf was the first to demonstrate this cancellation in 1934. . . well, actually he initially got it wrong, but thanks to Furry, the correct result was presented in an erratum.

The self-energy of the electron

V. WEISSKOPF

Zeitschrift für Physik, 89: 27–39 (1934). Received 13 March 1934.

The self-energy of the electron is derived in a closer formal connection with classical radiation theory, and the self-energy of an electron is calculated when the negative energy states are occupied, corresponding to the conception of positive and negative electrons in the Dirac 'hole' theory. As expected, the self-energy also diverges in this theory, and specifically to the same extent as in ordinary single-electron theory.

1 Problem definition

The self-energy of the electron is the energy of the electromagnetic field which is generated by the electron in addition to the energy of the interaction of the electron with this field. Waller,[1] Oppenheimer,[2] and Rosenfeld[3] calculated the self-energy of the free electron by means of the Dirac relativistic wave equation of the electron and the Dirac theory of the interaction between matter and light. They here used an approximation method which represents the self-energy in powers of the charge e. They found that the first term, which is proportional to e^2, already becomes infinitely large. The essential reason for this is that the theory of the interaction of the electron with the electromagnetic field is built on the classical equations of motion of a point-shaped electron whose self-energy, as is well known, also becomes infinite in classical theory.[4]

In the present note, the expressions for the self-energy shall be derived without direct application of quantum electrodynamics, but by means of the Heisenberg radiation theory,[5] which is linked much more closely to classical electrodynamics. The radiation field is calculated classically from the current and charge densities of the atom; however, the amplitudes of the electromagnetic potentials are regarded as non-commuting in the final result. Just as was shown in a corresponding paper by Casimir[6] concerning the natural linewidth, this method yields the same result as explicit quantum

[1] I. Waller, *ZS. f. Phys.* **62**, 673, 1930.
[2] R. Oppenheimer, *Phys. Rev.* **35**, 461, 1930.
[3] L. Rosenfeld, *ZS. f. Phys.* **70**, 454, 1931.
[4] Recently, G. Wentzel (*ZS. f. Phys.* **86**, 479, 635, 1933) has shown that one can circumvent the divergence of the self-energy in classical electron theory by suitable limiting processes. The transfer of these methods to quantum theory has failed, however, since, according to Waller, the degree of infinity in quantum theory is higher than in classical theory. The hope expressed there that the degree of infinity will become smaller in the Dirac formalism of the 'hole' theory, does indeed hold for the electrostatic part but not for the electrodynamic part, so that the Wentzel method must fail here too.
[5] W. Heisenberg, *Ann. d. Phys.* **338**, 1931; see also W. Pauli's article in Geiger-Scheel, *Handb. d. Phys.* XXIC/1, 2nd edn., pp. 201–10.
[6] H. Casimir, *ZS. f. Phys.* **81**, 496, 1933.

Correction to the paper: The self-energy of the electron

Zeitschrift für Physik, 90: 817–18 (1934). Received 20 July 1934.

On [p. 166] of the paper cited above, there is a computational error which has seriously garbled the results of the calculation for the electrodynamic self-energy of the electron according to the Dirac hole theory. I am greatly indebted to Mr Furry (University of California, Berkeley) for kindly pointing this out to me.

The degree of divergence of the self-energy in the hole theory is *not*, as asserted in [the preceding paper], just as great as in the Dirac one-electron theory, but the divergence is only logarithmic. The expression for the electrostatic and electrodynamic parts of the self-energy E of an electron with momentum p now correctly reads, in the notations used in [the preceding paper]:

$$E = E^{\mathrm{S}} + E^{\mathrm{D}},$$

$$E^{\mathrm{S}} = \frac{e^2}{h(m^2c^2 + p^2)^{1/2}}(2m^2c^2 + p^2)\int_{k_0}^{\infty}\frac{\mathrm{d}k}{k} + \text{finite terms},$$

$$E^{\mathrm{D}} = \frac{e^2}{h(m^2c^2 + p^2)^{1/2}}(m^2c^2 - \tfrac{4}{3}p^2)\int_{k_0}^{\infty}\frac{\mathrm{d}k}{k} + \text{finite terms}.$$

For comparison, we cite the expressions obtained on the basis of the single-electron theory:

$$E^{\mathrm{S}} = \frac{c^2}{h}\int_0^{\infty}\mathrm{d}k + \text{finite terms},$$

$$E^{\mathrm{D}} = \frac{e^2}{h}\left[\frac{m^2c^2}{p(m^2c^2 + p^2)^{1/2}}\log\frac{(m^2c^2 + p^2)^{1/2} + p}{(m^2c^2 + p^2)^{1/2} - p} - 2\right]\int_0^{\infty}\mathrm{d}k + \frac{2e^2}{h(m^2c^2 + p^2)^{1/2}}\int_0^{\infty}k\,\mathrm{d}k.$$

The computational error arose in the transformation of the electrodynamic portion E^{D} for the case of the hole theory:

$$E^{\mathrm{D}} = J_+^k(\vec{p}) - J_-^{\prime k}(\vec{p}), \quad k = 1 \text{ or } 2,$$

where $J_+^k(\vec{p})$ is defined on [p. 166] whereas

$$J_-^{\prime k}(\vec{p}) = -\frac{e^2}{2\pi h}\int\frac{\mathrm{d}\vec{k}}{k}\,\frac{PP_+ + \frac{1}{k^2}(\vec{k}\vec{p})^2 + (\vec{k}\vec{p}) + m^2c^2}{PP_+(P + P_+ + k)}$$

and is not equal to the quantity $J_-^k(\vec{p})$, from which it differs only by a sign. Likewise, one must set

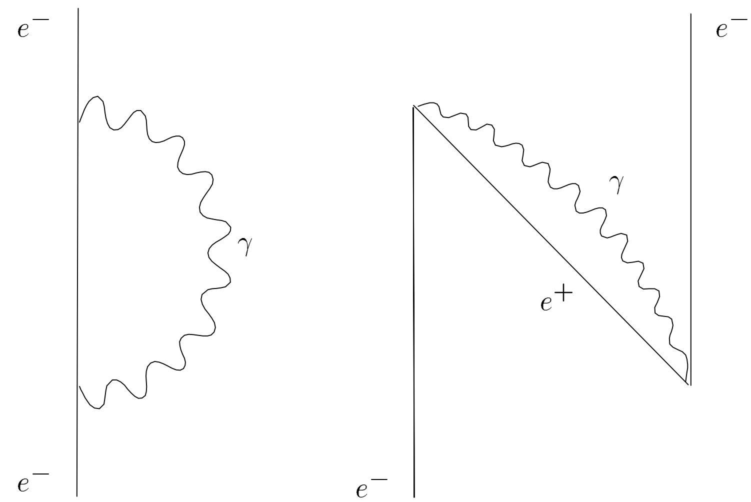
$$E_{\mathrm{vac}}^{\mathrm{D}} = \sum_{k=1,2}\int J_-^{\prime k}(\vec{p})\,\mathrm{d}\vec{p}$$

for the self-energy of the vacuum.

As a consequence of the new result, the question raised in note 4 of the paper requires a new examination, whether the Wentzel method,[15] to avoid the infinite self-energy by suitable limiting processes, might not still lead to the objective in the hole theory.

[15] G. Wentzel, *ZS. f. Phys*, 86, 479, 635, 1933.

A remarkable result:



(a) (b)

> The linear and quadratic divergences of a quantum theory of elementary fermions are precisely canceled if one doubles the particle spectrum—for every fermion, introduce an anti-fermion partner of the same mass and opposite charge.

In the process, we have introduced a new CPT-symmetry that associates a fermion with its anti-particle and guarantees the equality of their masses.

Low-Energy Supersymmetry

Will history repeat itself? Let's try it again. Take the Standard Model and double the particle spectrum. Introduce a new symmetry—supersymmetry—that relates fermions to bosons: for every fermion, there is a boson of equal mass and vice versa. Now, compute the self-energy of an elementary scalar. Supersymmetry relates it to the self-energy of a fermion, which is only logarithmically logarithmically sensitive to the fundamental high energy scale. Conclusion: quadratic sensitivity is removed! The hierarchy problem is resolved.

However, no super-partner (degenerate in mass with the corresponding SM particle) has ever been seen. Supersymmetry, if it exists in nature, must be a broken symmetry. Previous arguments then imply that:

> The scale of supersymmetry-breaking must be of order 1 TeV or less, if supersymmetry is associated with the scale of electroweak symmetry breaking.

Still to be understood—the origin of supersymmetry breaking [which is a difficult task not yet solved; there are many approaches but no compelling model]. Nevertheless, TeV-scale supersymmetric physics could provide our first glimpse of the Planck scale regime.

sensitivity to the ultraviolet (i.e. the largest energy scale) — another perspective

fermions in QFT

$$\delta m_\psi \sim g^2 m_\psi \ln \Lambda \qquad \text{due to mixing: } \psi_L \rightarrow \underset{m_\psi}{\times} \leftarrow \psi_R$$

is a consequence of chiral symmetry, which is exact if $m_f = 0$. Thus, δm_f must be proportional to m_f and by dimensional analysis is only logarithmically sensitive to Λ.

bosons in QFT

$$\delta m_\phi^2 \sim g^2 \Lambda^2$$

since there is no enhanced symmetry when $m_\phi = 0$. The mass term $m_\phi^2 \phi^* \phi$ is allowed by all continuous internal symmetries of the theory. By dimensional analysis, δm_ϕ^2 is proportional to Λ^2.

Thus, boson masses are not "protected" by internal symmetries.

bosons in supersymmetric QFT

If ϕ and ψ are supersymmetric particles in the same supermultiplet, then $m_\phi = m_\psi$ so that $\delta m_\phi = \delta m_\psi$.

Since $\delta m_\psi \sim g^2 m_\psi \ln \Lambda$, it follows that:

$$\delta m_\phi^2 = 2 m_\phi \delta m_\phi = 2 g^2 m_\phi^2 \ln \Lambda$$

Thus, fermion masses are protected by chiral symmetry while their bosonic superpartner masses are protected by supersymmetry.

II. Fermions in Quantum Field Theory

Some Technical Details

Supersymmetry is a bose-fermi symmetry. It will be very useful to understand and manipulate fermion fields at their most basic level.

We begin with some well known facts about the Poincaré algebra, which is the underlying space-time symmetry of relativistic quantum field theory.

$$[P^\mu, P^\nu] = 0$$

$$[J^{\mu\nu}, P^\lambda] = i(g^{\nu\lambda}P^\mu - g^{\mu\lambda}P^\nu)$$

$$[J^{\mu\nu}, J^{\lambda\rho}] = i(g^{\nu\lambda}J^{\mu\rho} - g^{\mu\lambda}J^{\nu\rho} + g^{\mu\rho}J^{\nu\lambda} - g^{\nu\rho}J^{\mu\lambda})$$

where P^μ generates space-time translations

$J^{\mu\nu}$ generates rotations and Lorentz boosts $\xrightarrow{\text{rot.}} J^i = \frac{1}{2}\epsilon^{ijk}J^k$, $\xrightarrow[\text{boosts}]{} K^i = J^{0i}$

The $J^{\mu\nu}$ satisfy an $SO(3,1) \simeq SL(2,\mathbb{C})$ Lie algebra.

In my conventions,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \epsilon_{0123} = +1.$$

The Pauli-Lubanski vector:

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma$$

$$W^0 = \vec{J}\cdot\vec{P}$$
$$\vec{W} = P^0\vec{J} + \vec{K}\times\vec{P}$$

Note that:

(i) $W_\mu P^\mu = 0$

(ii) $[W_\mu, P_\nu] = 0$

(iii) $[W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma$

Casimir operators of the Poincaré algebra:

(i) $P^2 = P_\mu P^\mu$

(ii) $W^2 = W_\mu W^\mu$

i.e. $[P^2, P^\mu] = [P^2, J^{\mu\nu}] = [W^2, P^\mu] = [W^2, J^{\mu\nu}] = 0$

Eigenvalues of P^2

$$P^2 = m^2$$

Eigenvalues of W^2

case 1: $m^2 > 0$. Go to rest frame where $P^\mu = (m; \vec{0})$
$\vec{J} = \vec{S}$

Then, $\vec{w} = m\vec{S}$

$w^2 = -m^2 \vec{S}^2$ with eigenvalues $-m^2 s(s+1)$, $s = 0, \frac{1}{2}, 1, \ldots$

case 2: $m^2 = 0$

Then $w^2 = 0$ [slight cheat here. But $w^2 = 0$ corresponds to the case of interest.]

$$w^2 = p^2 = w^\mu p_\mu = 0 \implies w^\mu = \lambda p^\mu$$

From $\vec{w} = p^0 \vec{J} + \vec{K} \times \vec{P}$,

$$\vec{w} \cdot \vec{P} = p^0 \vec{J} \cdot \vec{P}$$

For $p^2 = 0$, we have $p^0 = |\vec{P}|$. Thus, inserting $\vec{w} = \lambda \vec{P}$,

$$\lambda = \frac{\vec{J} \cdot \vec{P}}{|\vec{P}|} \qquad \text{helicity} \qquad |\lambda| = 0, \tfrac{1}{2}, 1, \ldots$$

Conclusion

1. Massive states of QFT characterized by $|m, s\rangle$
2. Massless states of QFT characterized by $|\lambda\rangle$.

Since λ changes sign under parity, when we add the antiparticle (the CPT-conjugate) the states are doubled: $|\lambda\rangle \oplus |-\lambda\rangle$. [example: the photon with $\lambda = \pm 1$.]

Fermions in QFT

Under a Lorentz transformation, $x'_\mu = L_\mu{}^\nu x_\nu$,

$$\psi'_\alpha(x') = \exp\left(-\frac{i}{2}\theta^{\mu\nu}S_{\mu\nu}\right)_\alpha{}^\beta \psi_\beta(x)$$

where the $S_{\mu\nu}$ are finite-dimensional matrices that satisfy the Lorentz algebra (same commutation relations as the $J_{\mu\nu}$).
Different choices for the $S_{\mu\nu}$ (irreducible representations) correspond to different spin. Clearly $S_{\mu\nu}=0$ corresponds to spin 0.

Define: $\quad J^i = \frac{1}{2}\varepsilon^{ijk}J_{jk} \qquad\Rightarrow\qquad [J^i,J^j] = i\varepsilon^{ijk}J^k$

$\quad K^i = J^{0i} \qquad\qquad [J^i,K^j] = i\varepsilon^{ijk}K^k$

$\qquad\qquad\qquad [K^i,K^j] = -i\varepsilon^{ijk}J^k$

and

$$\vec{J}_+ = \frac{1}{2}(\vec{J}+i\vec{K})$$
$$\vec{J}_- = \frac{1}{2}(\vec{J}-i\vec{K})$$

satisfy

$$[J_+^i, J_+^j] = i\varepsilon^{ijk}J_+^k$$
$$[J_-^i, J_-^j] = i\varepsilon^{ijk}J_-^k$$
$$[J_+^i, J_-^j] = 0$$

Thus, the irreducible representations of the Lorentz group correspond to (j_+, j_-), where the eigenvalues of $J_\pm^2$ are $j_\pm(j_\pm+1)$, respectively. The dimension of (j_+,j_-) is $(2j_++1)(2j_-+1)$.

example: $(0,0)$ is a scalar

Infinitesimally,

$$\exp\left(-\tfrac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\right) \simeq I - i\vec{\theta}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K}$$

$$\theta^i = \tfrac{1}{2}\epsilon^{ijk}\theta_{jk}$$
$$\beta^i = \theta^{0i}$$

<u>Two-dimensional</u> <u>(spin-½)</u> <u>representations</u>

$$\left(\tfrac{1}{2},0\right) \quad \left.\begin{aligned} \vec{J}_+ &= \tfrac{1}{2}(\vec{J}+i\vec{K}) = \frac{\vec{\sigma}}{2} \\ \vec{J}_- &= \tfrac{1}{2}(\vec{J}-i\vec{K}) = 0 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} \vec{J} &= \frac{\vec{\sigma}}{2} \\ \vec{K} &= -\frac{i\vec{\sigma}}{2} \end{aligned}$$

$$\left(0,\tfrac{1}{2}\right) \quad \left.\begin{aligned} \vec{J}_+ &= \tfrac{1}{2}(\vec{J}+i\vec{K}) = 0 \\ \vec{J}_- &= \tfrac{1}{2}(\vec{J}-i\vec{K}) = \frac{\vec{\sigma}}{2} \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} \vec{J} &= \frac{\vec{\sigma}}{2} \\ \vec{K} &= \frac{i\vec{\sigma}}{2} \end{aligned}$$

For the $(\frac{1}{2},0)$ representation, introduce the two-component field ξ_α ($\alpha=1,2$) which transforms under Lorentz transformations as:

$$\xi_\alpha \longrightarrow \xi'_\alpha = M_\alpha{}^\beta \xi_\beta$$

where $M \simeq I - \frac{i\theta\cdot\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}$ is a two-dimensional representation of $SL(2,C)$. In QFT, ξ_α is an anti-commuting two-component fermion field.

<u>Aside</u>: If M is a matrix representation of $SL(n,C)$, then M^*, $(M^{-1})^T$ and $(M^{-1})^\dagger$ are also representations, i.e. they preserve the group multiplication law. For $n>2$, all four representations are inequivalent. For $SL(2,C)$ only two of the four are distinct matrix representations for a given dimension, corresponding to (j_1,j_2) and (j_2,j_1).

For the $SL(2,C)$ matrices M, it is simple to check that:

$$(M^{-1})^T = i\sigma^2 M (i\sigma^2)^T \qquad i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which follows from:

$$\sigma^2 \vec{\sigma} \sigma^{2T} = \vec{\sigma}^T$$

Introduce the <u>contragradient</u> representation $(M^{-1})^T$:

$$\xi^\alpha \longrightarrow \xi'^\alpha = (M^{-1})^{T\,\alpha}{}_\beta\, \xi^\beta$$

$$= [i\sigma^2 M (i\sigma^2)^T]^\alpha{}_\beta\, \xi^\beta$$

which motivates the definition:

$$\varepsilon^{\alpha\beta} \equiv i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

i.e. $\varepsilon^{12} = -\varepsilon^{21} = 1$. Then,

$$\boxed{\xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta}$$

The matrices M and $(M^{-1})^T$ are related by similarity transformation, or equivalently by a change of basis; hence the corresponding representations are equivalent.

This is similar to the well-known result in $SU(2)$ that the $\underset{\sim}{2}$ and $\underset{\sim}{2}^*$ representations are equivalent.

Either ξ_α or ξ^α is a good candidate for the $(\frac{1}{2},0)$ representation.

For the $(0,\frac{1}{2})$ representation, introduce the "dotted" spinor indices:

$$\bar{\eta}^{\dot\alpha} \longrightarrow \bar{\eta}^{\dot\alpha\,\prime} = (M^{-1})^{\dagger\,\dot\alpha}{}_{\dot\beta}\,\bar{\eta}^{\dot\beta} \qquad \dot\alpha,\dot\beta = \dot1,\dot2$$

where

$$(M^{-1})^{\dagger} \simeq I - \frac{i\vec{\theta}\cdot\vec{\sigma}}{2} + \frac{\vec{\beta}\cdot\vec{\sigma}}{2}.$$

An equivalent description is via the <u>conjugate</u> representation M^*:

$$\bar{\eta}_{\dot\alpha} \longrightarrow \bar{\eta}_{\dot\alpha}' = (M^*)_{\dot\alpha}{}^{\dot\beta}\,\bar{\eta}_{\dot\beta}$$

where

$$\boxed{\bar{\eta}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\,\bar{\eta}_{\dot\beta}}$$

and:

$$\epsilon^{\dot\alpha\dot\beta} = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{\dot\alpha\dot\beta} = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So, $\epsilon^{\dot\alpha\dot\beta} = \epsilon^{\alpha\beta}$, etc.

Note that $\bar{\eta}_{\dot\alpha}$ and η_α^* have the same transformation law, so we may equate them:

$$\bar{\eta}_{\dot\alpha} = \eta_\alpha^*$$

Similarly,

$$\bar{\eta}^{\dot\alpha} = \eta^{\alpha\,*}.$$

Thus, ξ_α and $\bar{\eta}^{\dot{\alpha}}$ are the fundamental building blocks for constructing spin-½ quantum fields. To construct a field theory, we need to be able to construct Lorentz invariant scalar combinations of ξ and $\bar{\eta}$ in order to construct the Lagrangian.

A basic property of the Lorentz invariant matrix M is that:

$$\varepsilon^{\alpha\beta} M_\alpha{}^\rho M_\beta{}^\sigma \equiv \varepsilon^{\rho\sigma} \det M$$
$$= \varepsilon^{\rho\sigma}$$

It then follows that under
$$\xi_\alpha \to M_\alpha{}^\beta \xi_\beta$$
$$\chi_\alpha \to M_\alpha{}^\beta \chi_\beta$$

$$\chi\xi \equiv \chi^\alpha \xi_\alpha = \varepsilon^{\alpha\beta} \chi_\beta \xi_\alpha$$

is invariant under Lorentz transformations. Similarly,

$$\bar{\chi}\bar{\xi} \equiv \bar{\chi}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\chi}^{\dot{\beta}} \bar{\xi}^{\dot{\alpha}}$$

Note carefully the placement of the indices.

is invariant.

Notes:

1. $\chi\xi = \xi\chi$, using $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ and anti-commuting properties of the two-component fermion fields.
2. $\bar{\chi}\bar{\xi} = \bar{\xi}\bar{\chi}$
3. $(\chi\xi)^\dagger = (\chi^\alpha \xi_\alpha)^\dagger = \bar{\xi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\xi}\bar{\chi} = \bar{\chi}\bar{\xi}$

Hermitian conjugation reverses the order

Conclusion:

$$\chi\xi + \bar{\chi}\bar{\xi}$$

is Lorentz invariant and Hermitian. This is a candidate for a term in the Lagrangian.

We still need a candidate for a kinetic energy term.
Introduce:

$$\sigma^\mu = (I; \vec{\sigma})$$

$$\bar{\sigma}^\mu = (I, -\vec{\sigma})$$

Note that:

$$p_\mu \sigma^\mu = p_0 I - \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix}$$

is a Hermian 2×2 matrix. So is $M p_\mu \sigma^\mu M^\dagger$. Thus, there exists a p'_μ such that:

$$\boxed{p'_\mu \sigma^\mu = M p_\mu \sigma^\mu M^\dagger}$$

Exercise: Using $\det(p_\mu \sigma^\mu) = p_0^2 - |\vec{p}|^2$ and $\det M = 1$, show that $p_0'^2 - |\vec{p}'| = p_0^2 - |\vec{p}|^2$ and conclude that $p_\mu \to p'_\mu$ under the Lorentz transformation M.

The spinor index structure of the boxed equation above is:

$$p'_\mu \sigma^\mu_{\alpha\dot{\alpha}} = M_\alpha{}^\beta (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} p_\mu \sigma^\mu_{\beta\dot{\beta}}$$

Thus, we have deduced the spinor index structure of σ^μ:

$$\sigma^\mu_{\alpha\dot{\alpha}}$$

which immediately allows one to construct another Lorentz invariant quantity:

$$i\chi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\chi}^{\dot{\alpha}} \equiv i\chi\sigma^\mu\partial_\mu\bar{\chi}$$

The factor of i is inserted since $\frac{i}{2}\chi\sigma^\mu \overleftrightarrow{\partial_\mu}\bar{\chi}$ (which differs from $i\chi\sigma^\mu\partial_\mu\bar{\chi}$ by a total divergence) is hermitian and thus a candidate for a kinetic energy term in the Lagrangian.

<u>exercise</u>: Show that $(\chi\sigma^\mu\bar{\xi})^\dagger = \xi\sigma^\mu\bar{\chi}$.

Similarly, the index structure of $\bar{\sigma}^\mu$ is:

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}$$

<u>exercise</u>: Show that $\chi\sigma^\mu\bar{\xi} = -\bar{\xi}\bar{\sigma}^\mu\chi$.

That is, $\bar{\sigma}^\mu$ does not lead to an independent Lorentz invariant quantity.

<u>exercise</u>: Show that:

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu}$$
$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu}$$

Lorentz transformations in two component notation

$$\sigma^{\mu\nu}{}_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot\alpha}\,\bar\sigma^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\,\bar\sigma^{\mu\dot\alpha\beta}\right)$$

$$\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} = \frac{1}{4}\left(\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu{}_{\alpha\dot\beta} - \bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu{}_{\alpha\dot\beta}\right)$$

Explicitly,

$$\sigma^{ij} = -i\epsilon^{ijk}\frac{1}{2}\sigma^k = \bar\sigma^{ij}$$

$$\sigma^{i0} = -\sigma^{0i} = \frac{1}{2}\sigma^i = -\bar\sigma^{i0} = \bar\sigma^{0i}$$

note:

$$\epsilon_{\mu\nu\alpha\beta}\,\sigma^{\alpha\beta} = -2i\sigma_{\mu\nu}$$

$$\epsilon_{\mu\nu\alpha\beta}\,\bar\sigma^{\alpha\beta} = 2i\bar\sigma_{\mu\nu}$$

where $\epsilon_{0123} = +1$.

Comparing with

$$\exp -\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu} \simeq I - i\vec\theta\cdot\frac{\vec\sigma}{2} - \vec\beta\cdot\frac{\vec\sigma}{2}$$

$$\theta^i = \tfrac{1}{2}\epsilon^{ijk}\theta_{jk}$$

$$\beta^i = \theta^{i0} = -\theta^{0i}$$

we deduce that

$$\boxed{S^{\mu\nu} = i\sigma^{\mu\nu}}$$ for the $(\frac{1}{2},0)$ representation

Similarly,

$$\boxed{S^{\mu\nu} = i\bar\sigma^{\mu\nu}}$$ for the $(0,\frac{1}{2})$ representation.

Exercise: Show that:

$$\chi\sigma^{\mu\nu}\xi \equiv \chi^\alpha\sigma^{\mu\nu}{}_\alpha{}^\beta\xi_\beta = -\xi^\alpha\sigma^{\mu\nu}{}_\alpha{}^\beta\chi_\beta \equiv -\xi\sigma^{\mu\nu}\chi$$

$$\bar\chi\bar\sigma^{\mu\nu}\bar\xi \equiv \bar\chi_{\dot\alpha}\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta} = -\bar\xi_{\dot\alpha}\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}\bar\chi^{\dot\beta} \equiv -\bar\xi\bar\sigma^{\mu\nu}\bar\chi$$

$$(\chi\sigma^{\mu\nu}\xi)^\dagger = \bar\chi\bar\sigma^{\mu\nu}\bar\xi$$

Four-component notation

$$\psi = \begin{pmatrix} \xi_\alpha \\ \bar{\eta}^{\dot\alpha} \end{pmatrix}$$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_{\mu\alpha\dot\beta} \\ \bar{\sigma}_\mu^{\dot\alpha\beta} & 0 \end{pmatrix}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 2i \begin{pmatrix} \sigma^{\mu\nu}{}_\alpha{}^\beta & 0 \\ 0 & \bar{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} \end{pmatrix}$$

Note: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ $\qquad$ $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

Projection operators

$$P_L = \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_R = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

$$\psi_L \equiv P_L\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}$$

$$\psi_R \equiv P_R\psi = \begin{pmatrix} 0 \\ \bar{\eta}^{\dot\alpha} \end{pmatrix}$$

Dirac adjoint

$$\psi^\dagger = (\bar{\xi}_{\dot{\alpha}} \;\; \eta^\alpha)$$

Introduce the matrix A

$$A = \begin{pmatrix} 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \\ \delta_\alpha{}^\beta & 0 \end{pmatrix} \qquad A\gamma^\mu A^{-1} = \gamma^{\mu\dagger}$$

and the Dirac adjoint

$$\bar{\Psi} = \psi^\dagger A = (\eta^\beta \;\; \bar{\xi}_{\dot{\beta}})$$

remark: numerically, $A = \gamma^0$, although each has a different spinor-index structure.

Charge conjugation matrix

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \qquad C^{-1}\gamma^\mu C = -\gamma^{\mu T}$$

The charge conjugated four-component spinor is:

$$\psi^c = C\bar{\Psi}^T = C(\psi^\dagger A)^T = \begin{pmatrix} \eta_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}$$

remark: numerically, $C = i\gamma^0\gamma^2$, although each has a different spinor index structure.

Translation table relating bilinear covariants in two-component and four-component notation

$$\Psi_1(x) \equiv \begin{pmatrix} \xi_1(x) \\ \bar{\eta}_1(x) \end{pmatrix}, \qquad \Psi_2(x) \equiv \begin{pmatrix} \xi_2(x) \\ \bar{\eta}_2(x) \end{pmatrix}.$$

$\overline{\Psi}_1 P_L \Psi_2 = \eta_1 \xi_2$	$\overline{\Psi}{}^c_1 P_L \Psi^c_2 = \xi_1 \eta_2$
$\overline{\Psi}_1 P_R \Psi_2 = \bar{\xi}_1 \bar{\eta}_2$	$\overline{\Psi}{}^c_1 P_R \Psi^c_2 = \bar{\eta}_1 \bar{\xi}_2$
$\overline{\Psi}{}^c_1 P_L \Psi_2 = \xi_1 \xi_2$	$\overline{\Psi}_1 P_L \Psi^c_2 = \eta_1 \eta_2$
$\overline{\Psi}_1 P_R \Psi^c_2 = \bar{\xi}_1 \bar{\xi}_2$	$\overline{\Psi}{}^c_1 P_R \Psi_2 = \bar{\eta}_1 \bar{\eta}_2$
$\overline{\Psi}_1 \gamma^\mu P_L \Psi_2 = \bar{\xi}_1 \overline{\sigma}^\mu \xi_2$	$\overline{\Psi}{}^c_1 \gamma^\mu P_L \Psi^c_2 = \bar{\eta}_1 \overline{\sigma}^\mu \eta_2$
$\overline{\Psi}{}^c_1 \gamma^\mu P_R \Psi^c_2 = \xi_1 \sigma^\mu \bar{\xi}_2$	$\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = \eta_1 \sigma^\mu \bar{\eta}_2$
$\overline{\Psi}_1 \Sigma^{\mu\nu} P_L \Psi_2 = 2i\, \eta_1 \sigma^{\mu\nu} \xi_2$	$\overline{\Psi}{}^c_1 \Sigma^{\mu\nu} P_L \Psi^c_2 = 2i\, \xi_1 \sigma^{\mu\nu} \eta_2$
$\overline{\Psi}_1 \Sigma^{\mu\nu} P_R \Psi_2 = 2i\, \bar{\xi}_1 \overline{\sigma}^{\mu\nu} \bar{\eta}_2$	$\overline{\Psi}{}^c_1 \Sigma^{\mu\nu} P_R \Psi^c_2 = 2i\, \bar{\eta}_1 \overline{\sigma}^{\mu\nu} \bar{\xi}_2$

where, to avoid confusion, $\Sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Note that we may also write: $\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = -\eta_2 \overline{\sigma}^\mu \bar{\eta}_1$, *etc.* It then follows that:

$$
\begin{aligned}
&\overline{\Psi}_1\Psi_2 = \eta_1\xi_2 + \bar{\xi}_1\bar{\eta}_2\\
&\overline{\Psi}_1\gamma_5\Psi_2 = -\eta_1\xi_2 + \bar{\xi}_1\bar{\eta}_2\\
&\overline{\Psi}_1\gamma^\mu\Psi_2 = \xi_1\overline{\sigma}^\mu\xi_2 - \bar{\eta}_2\overline{\sigma}^\mu\eta_1\\
&\overline{\Psi}_1\gamma^\mu\gamma_5\Psi_2 = -\bar{\xi}_1\overline{\sigma}^\mu\xi_2 - \bar{\eta}_2\overline{\sigma}^\mu\eta_1\\
&\overline{\Psi}_1\Sigma^{\mu\nu}\Psi_2 = 2i(\eta_1\sigma^{\mu\nu}\xi_2 + \bar{\xi}_1\overline{\sigma}^{\mu\nu}\bar{\eta}_2)\\
&\overline{\Psi}_1\Sigma^{\mu\nu}\gamma_5\Psi_2 = -2i(\eta_1\sigma^{\mu\nu}\xi_2 - \bar{\xi}_1\overline{\sigma}^{\mu\nu}\bar{\eta}_2)\,.
\end{aligned}
$$

For Majorana fermions defined by $\Psi_M = \Psi_M^c = C\overline{\Psi}_M^T$, the following additional conditions are satisfied:

$$
\begin{aligned}
\overline{\Psi}_{M1}\Psi_{M2} &= \overline{\Psi}_{M2}\Psi_{M1}\,,\\
\overline{\Psi}_{M1}\gamma_5\Psi_{M2} &= \overline{\Psi}_{M2}\gamma_5\Psi_{M1}\,,\\
\overline{\Psi}_{M1}\gamma^\mu\Psi_{M2} &= -\overline{\Psi}_{M2}\gamma^\mu\Psi_{M1}\,,\\
\overline{\Psi}_{M1}\gamma^\mu\gamma_5\Psi_{M2} &= \overline{\Psi}_{M2}\gamma^\mu\gamma_5\Psi_{M1}\,,\\
\overline{\Psi}_{M1}\Sigma^{\mu\nu}\Psi_{M2} &= -\overline{\Psi}_{M2}\Sigma^{\mu\nu}\Psi_{M1}\,,\\
\overline{\Psi}_{M1}\Sigma^{\mu\nu}\gamma_5\Psi_{M2} &= -\overline{\Psi}_{M2}\Sigma^{\mu\nu}\gamma_5\Psi_{M1}\,.
\end{aligned}
$$

In particular, if $\Psi_{M1} = \Psi_{M2} \equiv \Psi_M$, then

$$
\overline{\Psi}_M\gamma^\mu\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\gamma_5\Psi_M = 0\,.
$$

Four-component Majorana spinor

Set $\eta = \xi$. Then,

$$\Psi_M = \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \xi \\ i\sigma^2 \xi^* \end{pmatrix}$$

One can check that $\Psi_M^c = \Psi_M$

Majorana field theory

$$\mathcal{L} = i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi})$$

$$= \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu \overleftrightarrow{\partial}_\mu \psi - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi}) + \text{total divergence}$$

where ψ is a two-component fermion.

<u>note</u>: $\bar{\psi}\overleftrightarrow{\partial}_\mu\psi \equiv \bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi$

Translating to four-component notation: $\quad \Psi_M \equiv \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$

$$\bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M = \bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\bar{\sigma}^\mu \psi$$

$$= \bar{\psi}\bar{\sigma}^\mu \overleftrightarrow{\partial}_\mu \psi$$

So that:

$$\mathcal{L} = \frac{i}{2}\bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - \frac{1}{2}m\bar{\Psi}_M\Psi_M$$

A Dirac fermion is equivalent to two mass-degenerate Majorana fermions.

Start with:

$$\mathcal{L} = i(\overline{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \overline{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2) - \tfrac{1}{2} m_{ij} \psi_i \psi_j - \tfrac{1}{2} m_{ij}^* \overline{\psi}_i \overline{\psi}_j$$

where $m_{ij} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$

and diagonalize m_{ij}. The corresponding eigenvalues are $\pm m$. Let:

$$\psi_a = \frac{\psi_1 + \psi_2}{\sqrt{2}} \qquad \text{or} \qquad \psi_1 = \frac{\psi_a + i\psi_b}{\sqrt{2}}$$

$$i\psi_b = \frac{\psi_1 - \psi_2}{\sqrt{2}} \qquad\qquad \psi_2 = \frac{\psi_a - i\psi_b}{\sqrt{2}}$$

the factor of i is inserted here so that both two-component fermions have positive mass.

Then,

$$\mathcal{L} = i(\overline{\psi}_a \bar{\sigma}^\mu \partial_\mu \psi_a + \overline{\psi}_b \bar{\sigma}^\mu \partial_\mu \psi_b)$$
$$- \tfrac{1}{2} m (\psi_a \psi_a + \overline{\psi}_a \overline{\psi}_a + \psi_b \psi_b + \overline{\psi}_b \overline{\psi}_b)$$

Corresponding to two mass-degenerate two-component spinors.

Dirac field theory

$$\mathcal{L} = i(\bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2) - \tfrac{1}{2} m_{ij} \psi_i \psi_j - \tfrac{1}{2} m_{ij}^* \bar{\psi}_i \bar{\psi}_j$$

where $m_{ij} \equiv \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$

and ψ_1, ψ_2 are both two-component spinors.

The Dirac spinor is

$$\psi_D = \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}$$

Note that:

$$\bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \bar{\sigma}^\mu \partial_\mu \psi_2 = \bar{\psi}_1 \bar{\sigma}^\mu \partial_\mu \psi_1 - (\partial_\mu \bar{\psi}_2) \bar{\sigma}^\mu \psi_2 + \text{total divergence}$$

$$= \bar{\psi}_D \gamma^\mu \partial_\mu \psi_D + \text{total divergence}$$

and

$$\psi_1 \psi_2 + \bar{\psi}_1 \bar{\psi}_2 = \bar{\psi}_D \psi_D$$

Thus,

$$\mathcal{L}_D = i \bar{\psi}_D \gamma^\mu \partial_\mu \psi_D - m \bar{\psi}_D \psi_D$$

Diagonalization of the fermion mass matrix

Consider n free anti-commuting two-component spin-½ fields, $\xi_{\alpha i}(x)$.

$i = 1, 2, \ldots, n$ is the flavor index.

The free-field Lagrangian is given by:

$$\mathcal{L} = i\,\bar{\xi}^i \bar{\sigma}^\mu \partial_\mu \xi_i - \tfrac{1}{2} M^{ij} \xi_i \xi_j - \tfrac{1}{2} M_{ij} \bar{\xi}^i \bar{\xi}^j$$

convention: $M_{ij} \equiv (M^{ij})^*$, i.e. raising or lowering flavor indices corresponds to complex conjugation.

The matrix M is complex and symmetric (since $\xi_i \xi_j = \xi_j \xi_i$).

To identify the physical masses, we must "diagonalize" M^{ij}. But, this is not a conventional diagonalization.

Define:

$$\xi_i = U_i{}^j \chi_j \,, \qquad \bar{\xi}^i = U^i{}_j \bar{\chi}^j \qquad U = \text{unitary matrix}$$

where $U^i{}_j = (U_i{}^j)^*$. The kinetic energy is unchanged but the mass term is transformed:

$$M^{ij} \xi_i \xi_j = M^{ij} U_i{}^k U_j{}^\ell \chi_k \chi_\ell = \sum_k m_k \chi_k \chi_k$$

with m_k real and non-negative, if

$$\boxed{U^T M U = M_D \equiv \mathrm{diag}(m_1, m_2, \ldots, m_n)}$$

This corresponds to the Takagi factorization of a general complex symmetric matrix.

Note: $U^\dagger M M^\dagger U = M_D^\dagger M_D = M_D^2$

However, U is not unique. Moreover, if degeneracies exist, one must work harder to determine U. For example, consider $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In terms of the mass eigenstates,

$$\mathcal{L} = i\bar{\chi}^i \bar{\sigma}^\mu \partial_\mu \chi_i - \tfrac{1}{2}\sum_i m_i (\chi_i \chi_i + \bar{\chi}^i \bar{\chi}^i).$$

If some of the m_i are equal, then the Lagrangian possesses an internal global symmetry. The most common case arises in a theory of charged fermions, which consists of a pair of mass degenerate two-component fermions.

Generalizing to a collection of free anti-commuting charged massive spin-½ fields, $\hat{\xi}_{\alpha i}(x)$ and $\hat{\eta}_{\alpha i}(x)$, where $\hat{\xi}$ and $\hat{\eta}$ are oppositely charged, the most general free-field Lagrangian is given by:

$$\mathcal{L} = i\bar{\hat{\xi}}^i \bar{\sigma}^\mu \partial_\mu \hat{\xi}_i + i\bar{\hat{\eta}}^i \bar{\sigma}^\mu \partial_\mu \hat{\eta}_i - M^{ij}\hat{\xi}_i \hat{\eta}_j - M_{ij}\bar{\hat{\xi}}^i \bar{\hat{\eta}}^j$$

Introduce the mass eigenstates

$$\hat{\chi}_i = L_i{}^k \chi_k, \qquad \hat{\eta}_i = R_i{}^k \eta_k$$

where L and R are unitary and satisfy:

$$\boxed{L^T M R = M_D = \text{diag}(m_1, m_2, \ldots)}$$

where the m_i are real and non-negative. This is called the singular value decomposition of an arbitrary complex matrix M.

Note:
$$L^T (M M^\dagger) L^* = M_D M_D^\dagger = M_D^2$$
$$R^\dagger (M^\dagger M) R = M_D^\dagger M_D = M_D^2$$

As before, these two equations may not be sufficient to determine L, R, although they do determine the m_i as the positive square roots of the diagonal elements of M_D^2.

In terms of the mass eigenstates,

$$\mathcal{L} = i\bar{\chi}^i \bar{\sigma}^\mu \partial_\mu \chi_i + i\bar{\eta}^i \bar{\sigma}^\mu \partial_\mu \eta_i - \sum_i m_i (\chi_i \eta_i + \bar{\chi}^i \bar{\eta}^i).$$

The see-saw mechanism

The see-saw Lagrangian is given by:

$$\mathcal{L} = i\left(\overline{\psi^1}\,\overline{\sigma}^\mu \partial_\mu \psi_1 + \overline{\psi^2}\,\overline{\sigma}^\mu \partial_\mu \psi_2\right) - M^{ij}\psi_i\psi_j - M_{ij}\overline{\psi^i}\,\overline{\psi^j}\,,$$

where

$$M^{ij} = \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix},$$

and (without loss of generality) m_D and M are positive. The Takagi factorization of this matrix is $U^T M U = M_D$, where

$$U = \begin{pmatrix} i\cos\theta & \sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}, \qquad M_D = \begin{pmatrix} m_- & 0 \\ 0 & m_+ \end{pmatrix},$$

and where $m_\pm = \frac{1}{2}\left[\sqrt{M^2 + 4m_D^2} \pm M\right]$ and

$$\sin 2\theta = \frac{2m_D}{\sqrt{M^2 + 4m_D^2}}, \qquad \cos 2\theta = \frac{M}{\sqrt{M^2 + 4m_D^2}}.$$

If $M \gg m_D$, then the corresponding fermion masses are $m_- \simeq m_D^2/M$ and $m_+ \simeq M$, while $\sin\theta \simeq m_D/M$. The mass eigenstates, χ_i are given by $\psi_i = U_i{}^j \chi_j$; *i.e.* to leading order in m_d/M,

$$
\begin{aligned}
i\chi_1 &\simeq \psi_1 - \frac{m_D}{M}\psi_2 \,, \\
\chi_2 &\simeq \psi_2 + \frac{m_D}{M}\psi_1 \,.
\end{aligned}
$$

Indeed, one can check that:

$$
\begin{aligned}
&\tfrac{1}{2} m_D(\psi_1\psi_2 + \psi_2\psi_1) + \tfrac{1}{2} M\psi_2\psi_2 + \text{h.c.} \\
&\qquad \simeq \tfrac{1}{2}\left[\frac{m_D^2}{M}\chi_1\chi_1 + M\chi_2\chi_2 + \text{h.c.}\right] \,,
\end{aligned}
$$

which corresponds to a theory of two Majorana fermions—one very light and one very heavy (the see-saw).

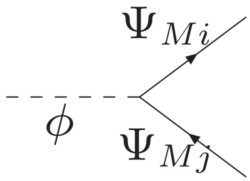
Feynman rules for Majorana fermions

Consider a set of neutral and charged fermions interacting with a neutral scalar or vector boson. The interaction Lagrangian in terms of two-component fermions is:

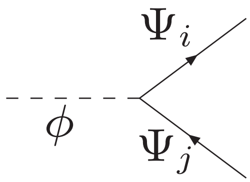
$$\mathcal{L}_{\rm int} = -\tfrac{1}{2}(\lambda^{ij}\xi_i\xi_j + \lambda_{ij}\bar{\xi}^i\bar{\xi}^j)\phi - (\kappa^{ij}\chi_i\eta_j + \kappa_{ij}\bar{\chi}^i\bar{\eta}^j)\phi$$

$$-(G_\xi)_i{}^j\,\bar{\xi}^i\overline{\sigma}^\mu\xi_j A_\mu - [(G_\chi)_i{}^j\bar{\chi}^i\overline{\sigma}^\mu\chi_j + (G_\eta)_i{}^j\bar{\eta}^i\overline{\sigma}^\mu\eta_j]A_\mu\,,$$

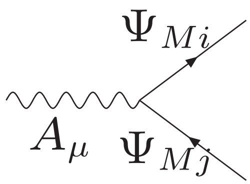
where λ is a complex symmetric matrix, κ is an arbitrary complex matrix and G_ξ, G_χ and G_η are hermitian matrices. By assumption, χ and η have the opposite U(1) charges, while all other fields are neutral.



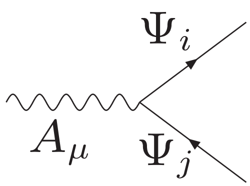
$$-i(\lambda^{ij}P_L + \lambda_{ij}P_R)$$



$$-i(\kappa^{ji}P_L + \kappa_{ij}P_R)$$



$$-i\gamma_\mu[(G_\xi)_i{}^jP_L - (G_\xi)_j{}^iP_R]$$



$$-i\gamma_\mu[(G_\chi)_i{}^jP_L - (G_\eta)_j{}^iP_R]$$

The arrows on the Dirac fermion lines depict the flow of the conserved charge. A Majorana fermion is self-conjugate, so its arrow simply reflects the structure of $\mathcal{L}_{\rm int}$; *i.e.*, $\overline{\Psi}_M$ [Ψ_M] is represented by an arrow pointing out of [into] the vertex. The arrow directions determine the placement of the u and v spinors in an invariant amplitude.

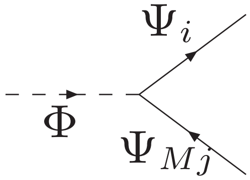
Next, consider the interaction of fermions with charged bosons, where the charges of Φ, W and χ are assumed to be equal. The corresponding interaction Lagrangian is given by:

$$\begin{aligned}\mathcal{L}_{\rm int} = &-\tfrac{1}{2}\Phi^*[\kappa_1^{ij}\chi_i\xi_j + (\kappa_2)_{ij}\bar{\eta}^i\bar{\xi}^j] - \tfrac{1}{2}\Phi[\kappa_2^{ij}\eta_i\xi_j + (\kappa_1)_{ij}\bar{\chi}^i\bar{\xi}^j] \\ &-\tfrac{1}{2}W_\mu[(G_1)_i{}^j\bar{\chi}^i\overline{\sigma}^\mu\xi_j + (G_2)_i{}^j\bar{\xi}_i\overline{\sigma}^\mu\eta_j] \\ &-\tfrac{1}{2}W_\mu^*[(G_1)_j{}^i\bar{\xi}^j\overline{\sigma}^\mu\chi_i + (G_2)_j{}^i\bar{\eta}^j\overline{\sigma}^\mu\xi_i]\,,\end{aligned}$$

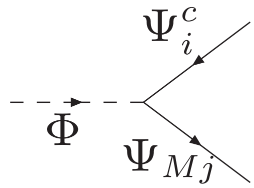
where κ_1 and κ_2 are complex symmetric matrices and G_1 and G_2 are hermitian matrices. We now convert to four-component spinors, and note that $C^T = -C$ and anti-commuting fermion fields imply that

$$\overline{\Psi}{}^c_i\Gamma\Psi^c_j = -\Psi_i^T C^{-1}\Gamma C\overline{\Psi}{}_j^T = \overline{\Psi}_j C\Gamma^T C^{-1}\Psi_i = \eta_\Gamma\overline{\Psi}_j\Gamma\Psi_i\,,$$

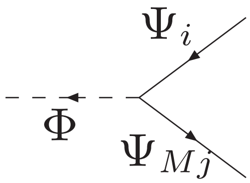
where the sign $\eta_\Gamma = +1$ for $\Gamma = 1, \gamma_5, \gamma^\mu\gamma_5$ and $\eta_\Gamma = -1$ for $\Gamma = \gamma^\mu$, $\Sigma^{\mu\nu}$, $\Sigma^{\mu\nu}\gamma_5$. Hence, the Feynman rules for the interactions of neutral and charged fermions with charged bosons can take two possible forms:



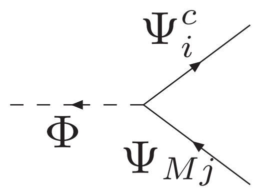
or



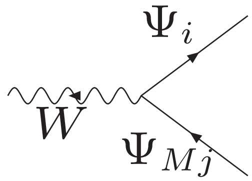
$$-i(\kappa^{2ij}P_L + \kappa_{1ij}P_R)$$



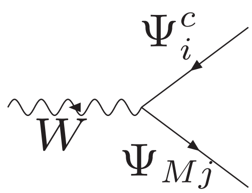
or



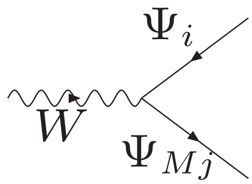
$$-i(\kappa^{1ij}P_L + \kappa_{2ij}P_R)$$



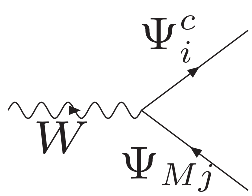
$$-i\gamma^\mu({G_1}_i{}^j P_L - {G_2}_j{}^i P_R)$$



$$i\gamma^\mu({G_1}_i{}^j P_R - {G_2}_j{}^i P_L)$$



$$-i\gamma^\mu({G_1}_j{}^i P_L - {G_2}_i{}^j P_R)$$



$$i\gamma^\mu({G_1}_i{}^j P_R - {G_2}_j{}^i P_L)$$

One is free to choose either a Ψ or Ψ^c line to represent a Dirac fermion at any place in a given Feynman graph. The direction of the arrow on the Ψ or Ψ^c line indicates the corresponding direction of charge flow.[1]

[1] Since the charge of Ψ^c is opposite to that of Ψ, the corresponding arrow direction of the two lines point in opposite directions.

Moreover, the structure of $\mathcal{L}_{\rm int}$ implies that the arrow directions on fermion lines flow continuously through the diagram. This requirement then determines the direction of the arrows on Majorana fermion lines. In the computation of a given process, one may employ either Ψ or Ψ^c when representing the propagation of a (virtual) Dirac fermion. Because free Dirac fields satisfy:

$$\left\langle 0|T(\Psi_\alpha(x)\overline{\Psi}_\beta(y))|0\right\rangle = \left\langle 0|T(\Psi^c_\alpha(x)\overline{\Psi}^c_\beta(y))|0\right\rangle ,$$

the Feynman rules for the propagator of a Ψ and Ψ^c line are identical.

Construction of invariant amplitudes involving Majorana fermions

When computing an invariant amplitude, one first writes down the relevant Feynman diagrams with no arrows on any Majorana fermion line. The number of distinct graphs contributing to the process is then determined. Finally, one makes some choice for how to distribute the arrows on the Majorana fermion lines and how to label Dirac fermion lines (either Ψ or Ψ^c) in a manner consistent with the Feynman rules for the interaction vertices. The end result for the invariant amplitude (apart from an overall unobservable phase) does not depend on the choices made for the direction of the fermion arrows.

Using the above procedure, the Feynman rules for the external fermion wave functions are the same for Dirac and Majorana fermions:

- $u(\vec{\boldsymbol{p}}, s)$: incoming Ψ [or Ψ^c] with momentum $\vec{\boldsymbol{p}}$ parallel to the arrow direction,
- $\bar{u}(\vec{\boldsymbol{p}}, s)$: outgoing Ψ [or Ψ^c] with momentum $\vec{\boldsymbol{p}}$ parallel to the arrow direction,
- $v(\vec{\boldsymbol{p}}, s)$: outgoing Ψ [or Ψ^c] with momentum $\vec{\boldsymbol{p}}$ anti-parallel to the arrow direction,
- $\bar{v}(\vec{\boldsymbol{p}}, s)$: incoming Ψ [or Ψ^c] with momentum $\vec{\boldsymbol{p}}$ anti-parallel to the arrow direction.

Example: $\Psi(p_1)\Psi(p_2) \longrightarrow \Phi(k_1)\Phi(k_2)$ via Ψ_M-exchange

The contributing Feynman graphs are:

Following the arrows in reverse, the resulting invariant amplitude is:

$$i\mathcal{M} = (-i)^2 \bar{v}(\vec{\boldsymbol{p}}_2, s_2)(\kappa_1 P_L + \kappa_2^* P_R) \left[\frac{i(\not{p}_1 - \not{k}_1 + m)}{t - m^2} + \frac{i(\not{k}_1 - \not{p}_2 + m)}{u - m^2} \right] (\kappa_1 P_L + \kappa_2^* P_R) u(\vec{\boldsymbol{p}}_1, s_1) \,,$$

where $t \equiv (p_1 - k_1)^2$, $u \equiv (p_2 - k_1)^2$ and m is the Majorana fermion mass. The sign of each diagram is determined simply by the relative permutation of spinor factors appearing in the amplitude (the overall sign of the amplitude is unphysical).

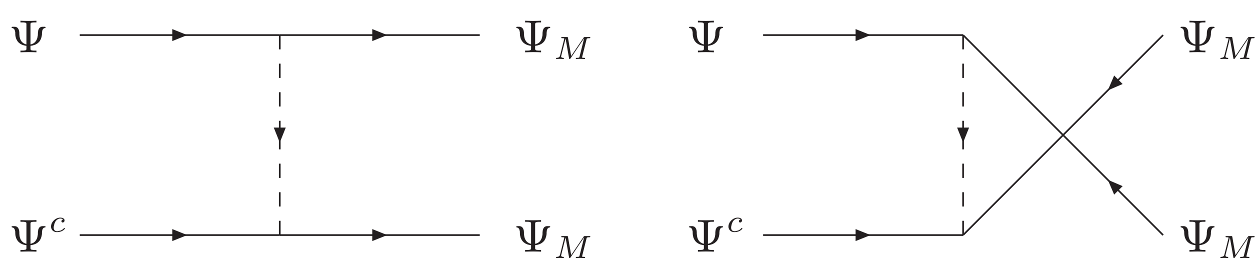
Exercise: Check that $i\mathcal{M}$ is antisymmetric under interchange of the two initial electrons. *HINT:* Taking the transpose and using $v \equiv u^c \equiv C\bar{u}^T$ (the u and v spinors are commuting objects), one easily verifies that:

$$\bar{v}(\vec{\boldsymbol{p}}_2, s_2)\Gamma u(\vec{\boldsymbol{p}}_1, s_1) = -\eta_\Gamma \bar{v}(\vec{\boldsymbol{p}}_1, s_1)\Gamma u(\vec{\boldsymbol{p}}_2, s_2)\,,$$

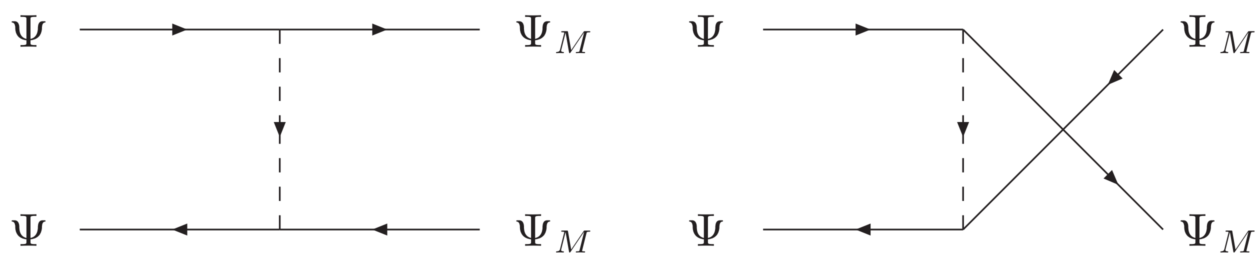
where as before $\eta_\Gamma = +1$ for $\Gamma = 1, \gamma_5, \gamma^\mu\gamma_5$ and $\eta_\Gamma = -1$ for $\Gamma = \gamma^\mu$, $\Sigma^{\mu\nu}$, $\Sigma^{\mu\nu}\gamma_5$.

Example: $\Psi(p_1)\Psi^c(p_2) \to \Psi_M(p_3)\Psi_M(p_4)$ via charged Φ-exchange

Neglecting a possible s-channel annihilation graph, the contributing Feynman graphs can be represented either by diagram set (i):



or by diagram set (ii):

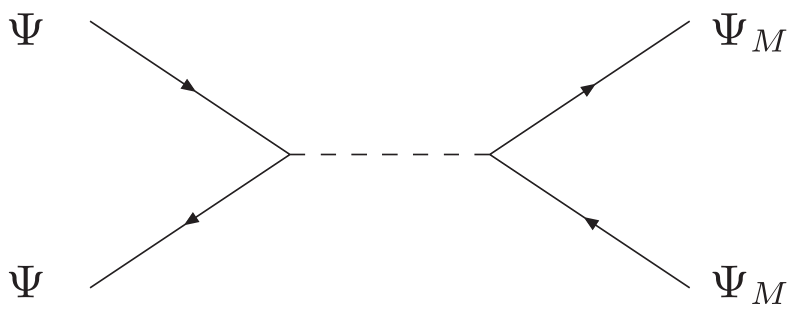


The amplitude is evaluated by following the arrows in reverse. Using:

$$\bar{v}(\vec{\boldsymbol{p}}_2, s_2)\Gamma v(\vec{\boldsymbol{p}}_4, s_4) = -\eta_\Gamma \bar{u}(\vec{\boldsymbol{p}}_4, s_4)\Gamma u(\vec{\boldsymbol{p}}_2, s_2)\,,$$

one can check that the invariant amplitudes resulting from diagram sets (i) and (ii) differ by an overall minus sign, as expected due to the fact that the corresponding order of the spinor wave functions differs by an odd permutation [$e.g.$, for the t-channel graphs, compare 3142 and 3124 for (i) and (ii) respectively]. For the same reason, there is a relative minus sign between the t-channel and u-channel graphs for either diagram set [$e.g.$, compare 3142 and 4132 in diagram set(i)].

If s-channel annihilation contributes, its calculation is straightforward:



Relative to the t-channel graph of diagram set (ii), this diagram comes with an extra minus sign [since 2134 is odd with respect to 3124].

In the computation of the unpolarized cross-section, non-standard spin projection operators can arise in the evaluation of the interference terms. One may encounter spin sums such as:[2]

$$\sum_s u(\vec{\boldsymbol{p}}, s) v^T(\vec{\boldsymbol{p}}, s) = (\not{p} + m) C^T \,,$$

$$\sum_s \bar{u}^T(\vec{\boldsymbol{p}}, s) \bar{v}(\vec{\boldsymbol{p}}, s) = C^{-1}(\not{p} - m) \,,$$

which requires additional manipulation of the charge conjugation matrix C. However, these non-standard spin projection operators can be avoided by judicious use of spinor product relations of the kind displayed on the previous two pages.

[2] see Appendix D of G.L. Kane and H.E. Haber, *Phys. Rep.* **117** (1985) 75.