Introduction to Superstrings

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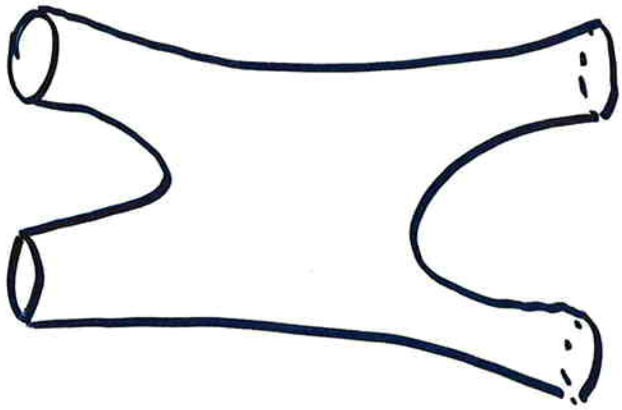
0. What is string theory?

Essential idea: fundamental object out of which everything is made are strings

closed open

In terms of the string point of view elementary particles correspond to the different vibration modes of the string.

Interactions of strings is geometrical: joining & splitting of strings



no singular interaction vertex: better UV behaviour than usual field theories.

Also, as we shall see, string spectrum contains graviton, and string theory therefore incorporates gravity into elementary particle physics.

Plan of lecture

- classical string equations
  and boundary conditions

- quantisation (light-cone gauge)
  and description of spectrum

- Compactification and T-duality

(• orbifolds)

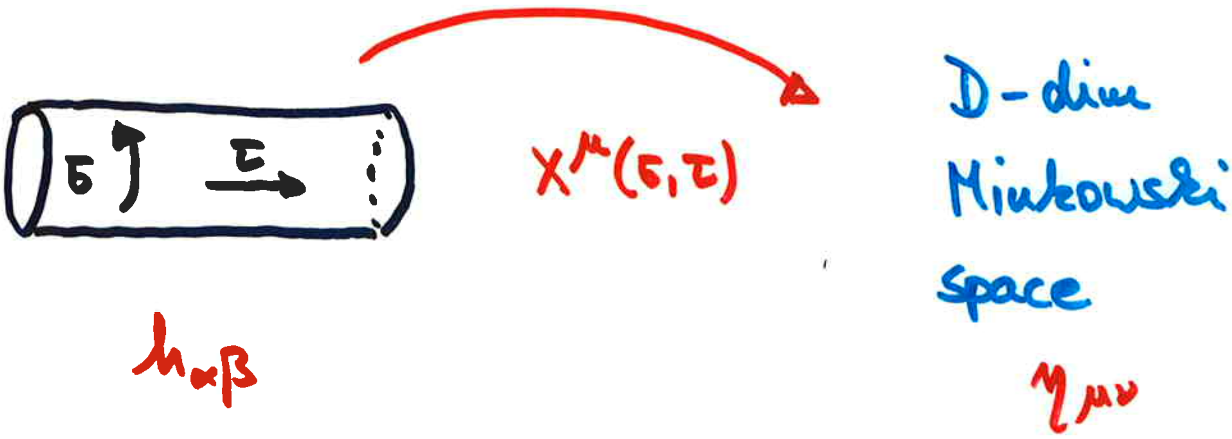
1. Bosonic string theory

Bosonic string that propagates in D-dim Minkowski space is described by Polyakov action

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, \eta_{\mu\nu}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu$$

string tension

Here $h_{\alpha\beta}$ is the metric on the 2-dim worldsheet of string



This action is invariant under general coordinate transformation of worldsheet

$$(\tau, \sigma) \mapsto \left(\tilde{\tau}(\tau,\sigma),\ \tilde{\sigma}(\tau,\sigma)\right)$$

From point of view of world-sheet the X^μ are scalar fields, but they transform as vectors under the (global) Poincaré group of D-dim Minkowski space.

By using the invariance under reparametrisation of world-sheet can go to conformal gauge

$$h_{\alpha\beta} = e^{\phi}\,\eta_{\alpha\beta}$$

Since $\sqrt{h} \sim e^{\phi}$ and $h^{\alpha\beta} \sim e^{-\phi}$, conformal factor e^{ϕ} drops out, and action becomes

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma\ \eta^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu$$

The equations of motion derived from this action are simply linear wave equations

$$\Box X^\mu = \left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu = 0 .$$

Since they have been derived from gauge fixed action, they have to be supplemented by constraint equations

$$T_{\alpha\beta} = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h^{\alpha\beta}} = 0$$

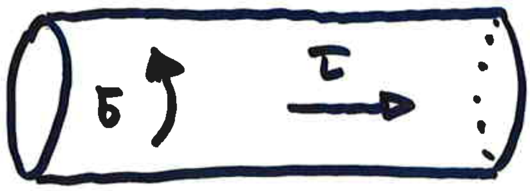
If we define $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ this becomes the Virasoro constraint

$$T_{--} = -\frac{1}{\alpha'}\, \partial_- X^\mu \, \partial_- X_\mu = 0$$

$$T_{++} = -\frac{1}{\alpha'}\, \partial_+ X^\mu \, \partial_+ X_\mu = 0$$

Before solving these equations need to discuss boundary conditions.

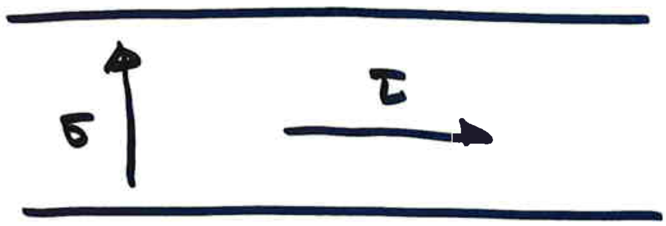
For closed strings, worldsheet is cylinder



$\sigma \longmapsto \sigma + 2\pi$

periodic

For open strings the worldsheet is infinite strip



$0 \leq \sigma \leq \pi$

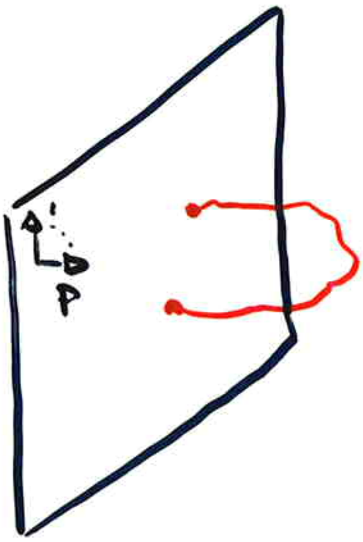
and the boundary conditions follow from the variation of the action

$$\delta S = \text{bulk terms} - \int \frac{d\tau}{\pi\alpha'} \, \delta X^\mu \, \partial_\sigma X_\mu \Big|_{\sigma=0}^{\sigma=\pi}$$

The boundary terms vanish if at $\sigma=0$ and $\sigma=\pi$

$$\boxed{\begin{array}{ll} \partial_\sigma X^\mu = 0 & \text{NEUMANN} \\ \text{or} \quad X^\mu = 0 \text{ (fixed)} & \text{DIRICHLET} \end{array}}$$

Let us choose Neumann conditions for $X^{0,1,\dots,p}$ and Dirichlet conditions for $X^{p+1,\dots,D-1}$. (Here $p \leq D-1$.) In modern language this then describes the open string whose endpoints lie on a Dirichlet p-brane



open string oscillates in the (D-1)-dim bulk but its endpoints are attached to p-dim brane

The mode expansions of this open string is then

$$X^\mu = \underbrace{x^\mu + 2\alpha' p^\mu \tau}_{\text{for Neumann only}} +$$

$$+ i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \begin{cases} \sin(n\sigma) & D \\ \cos(n\sigma) & N \end{cases}$$

Furthermore, the gauge conditions are best expressed in terms of the Fourier modes of T_{--}

$$0 = L_N \equiv \frac{1}{2} \sum_n \alpha^\mu_{N-n} \alpha_{\mu n}$$

$$[\alpha_0^\mu \equiv \sqrt{2\alpha'} p^\mu]$$

On the other hand, the closed string has the mode expansion

$$X^\mu(\tau,\sigma) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$$

with

$$X_R^\mu = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau-\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau+\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}$$

Furthermore the gauge conditions are

$$0 = L_N = \frac{1}{2}\sum_n \alpha_{N-n}^\mu \alpha_{\mu n}$$

$$0 = \tilde{L}_N = \frac{1}{2}\sum_n \tilde{\alpha}_{N-n}^\mu \tilde{\alpha}_{\mu n}.$$

1.1 Quantisation

Let us first discuss quantisation of open string. [Analysis for closed string essentially the same since

$$\text{closed} \sim (\text{open})_L \cdot (\text{open})_R .]$$

Quantisation can be performed by imposing canonical commutation relations either before (covariant) or after (light-cone) solving the gauge constraints.

These gauge constraints reflect the fact that conformal gauge

$$h_{\alpha\beta} = e^{\phi}\, \eta_{\alpha\beta}$$

does not fix reparametrisation invariance completely. In fact, we still have the conformal transformations

$$\tau + \sigma \mapsto f(\tau+\sigma) \qquad \tau - \sigma \mapsto \tilde{f}(\tau-\sigma)$$

where f and $\tilde{f}$ are arbitrary functions.

This redundancy can be fixed

by choosing a light-cone coordinate

$$X^+ \equiv \frac{X^0 + X^1}{\sqrt{2}} \qquad (\text{assuming } p \geq 1)$$

decomposing it into a left- and a

right-moving field, $X^+ = X^+_L(\tau+\sigma) + X^+_R(\tau-\sigma)$

and then choosing

$$f = X_L / \alpha' p^+ \qquad \tilde{f} = X_R / \alpha' p^+$$

[This maps strip to strip!]

Then

$$X^+ = 2\alpha' p^+ \tau$$

In particular, we therefore have

$$\alpha_n^+ = 0 \quad \text{for} \quad n \neq 0.$$

Defining $X^- = \dfrac{X^0 - X^1}{\sqrt{2}}$ and

observing that

$$A^\mu B_\mu = -A^+B^- - A^-B^+ + \sum_{j=2}^{D-1} A^j B^j$$

one can then also solve the gauge conditions for α_n^- (from $L_N = 0$).

The independent variables are thus the transverse oscillators

$$\alpha_n^j \qquad j = 2, 3, \ldots, D-1$$

together with p^+, x^- and the x^j.

[Here $\alpha_0^j \equiv \sqrt{2\alpha'}\, p^j$.]

The $L_0 = 0$ constraint gives in particular (this fixes p^-)

$$\alpha' M^2 = \sum_{\substack{n>0 \\ j}} \alpha_n^j \alpha_{-n}^j$$

'mass shell condition'

$$M^2 = p^\mu p_\mu$$

In light-cone quantisation one then imposes canonical commutation relations for these degrees of freedom:

$$[\alpha^i_m, \alpha^l_n] = m\, \delta^{il}\, \delta_{m,-n}$$

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}$$

As in standard field theory the states of the theory are then built on a 'vacuum' $|0,p\rangle$ (unexcited state of string with 'centre of mass' momentum p)

with

$$\alpha^j_n \, |0, p\rangle = 0 \qquad n > 0$$

$$\alpha^\mu_0 \, |0, p\rangle = \sqrt{2\alpha'} \, p^\mu \, |0, p\rangle .$$

The excited states are obtained by acting with the raising operators α^j_n with $n < 0$.

The definition of L_N with $N \neq 0$ is unambiguous, but for

$$L_0 = \frac{1}{2} \sum_{\substack{n \neq 0 \\ j}} \alpha^j_n \, \alpha^j_{-n} + \alpha' p^\mu p_\mu$$

there is a normal ordering ambiguity, which affects mass shell condition.

To fix this ambiguity, analyse constraint that requires that light-cone gauge theory has full D-dim Poincaré symmetry, in particular

$$[\, J^{i-}, J^{j-}] = 0 .$$

This fixes then mass-shell condition to be

$$\alpha' M^2 = N - 1 ,$$

where $N = \sum_{\substack{n=1 \\ j}}^{\infty} \alpha^j_{-n} \alpha^j_n$ number operator

$$[\, N \alpha^{j_1}_{-n_1} \cdots \alpha^{j_\ell}_{-n_\ell} |0,p\rangle = (n_1 + \dots + n_\ell) \cdot \alpha^{j_1}_{-n_1} \cdots \alpha^{j_\ell}_{-n_\ell} |0,p\rangle . \,]$$

Furthermore, D-dim Poincaré-symmetry implies that

$$D = 26 \quad - \text{ 'critical dimension'}$$

As an aside, these constraints can also be obtained in covariant quantisation — there D-dim Poincaré symmetry is manifest but one needs to require that physical spectrum does not have negative norm states (ghost): 'No-ghost theorem'.

Now can describe spectrum of open string:

N=0 : these are the states $|0,p\rangle$

Mass shell condition gives

$$M^2 = p^\mu p_\mu = -\frac{1}{\alpha'} \qquad \text{tachyon}$$

N=1 : these are the states $\alpha^j_{-1}|0,p\rangle$

Mass shell condition gives

$$M^2 = 0 \qquad \text{massless states}$$

1-particle states of massless vector field A^μ, reduced from D=26 to p+1 dimensions.

$N=2$: these are the states

$$\alpha^i_{-1}\,\alpha^j_{-1}\,|0,p\rangle \quad \text{and} \quad \alpha^j_{-2}\,|0,p\rangle$$

Now mass shell condition gives

$$M^2 = \frac{1}{\alpha'}$$

massive states
(mass ~ string scale)

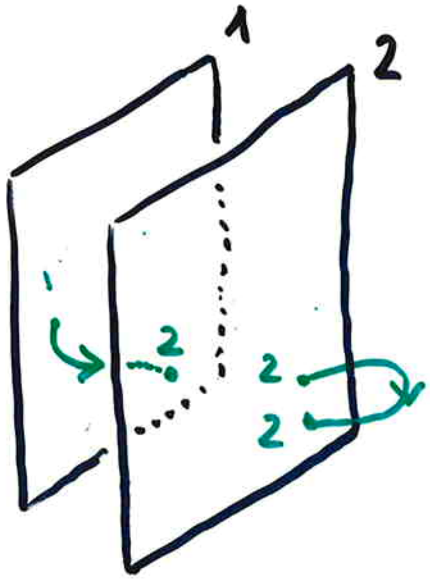
There are

$$\frac{1}{2}(D-2)(D-1) + (D-2) = \frac{1}{2}(D-2)(D+1)$$

such states — lie in symmetric traceless representation of little group $SO(D-1)$

etc.

We can make things more interesting by putting n Dp-branes together



Oriented strings (and the corresponding fields) can now be labelled by an ordered pair of Chan-Paton indices

The mass spectrum of the open string from a to b is then

$$\alpha' M^2 = \frac{1}{4\pi^2\alpha'} \, | \vec{r}_a - \vec{r}_b |^2 + N - 1$$

position in transverse space

If all the D-branes sit at the same position in transverse space, we get n^2 massless vector fields (reduced from $D=26$ to $p+1$). In fact, these vector fields transform in the adjoint representation of $U(n)$:

gauge enhancement $U(1)^n \rightarrow U(n)$.

Conversely, different positions of the D-branes correspond to the Coulomb branch of the gauge theory.

The analysis for closed strings (that always appear, even in a 'theory of open strings') is similar. Now one has two sets of oscillators

$$\alpha^j_n \quad \text{and} \quad \tilde{\alpha}^j_n$$

and mass shell condition is

$$\alpha' M^2 = 4(N-1) = 4(\tilde{N}-1).$$

In particular, we have the level matching condition

$$N = \tilde{N}.$$

The low-lying spectrum of the closed string is thus

$N=0$: ground state tachyon $|0,p\rangle$

with

$$M^2 = -\frac{4}{\alpha'} .$$

$N=1$: massless states of the form

$$\alpha^i_{-1} \, \tilde{\alpha}^j_{-1} \, |0,p\rangle$$

with

$$M^2 = p^\mu p_\mu = 0$$

In terms of representations of the little group $SO(24)$, these decompose as

- symmetric traceless part
  describes massless spin 2
  particle: graviton

- trace term is massless scalar:
  dilaton [~ string coupling]

- antisymmetric part describes
  antisymmetric 2nd rank tensor:
  Kalb-Ramond field (B-field)

+ massive states (string scale) ...

2. Adding supersymmetry

The bosonic string we have discussed so far has a tachyon (D=26 Minkowski space is 'wrong vacuum'), and does not describe spacetime fermions. In order to overcome these limitations, consider supersymmetric generalisation of bosonic string. In the following we want to discuss the NS-R superstring.

[Neveu-Schwarz, Ramond]

To this end we add to the above bosonic action (in conformal gauge, say) worldsheet (Majorana) fermions

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left[\partial_\alpha X^\mu \partial^\alpha X_\mu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right]$$

Here ρ^α are 2d Dirac matrices, which can be chosen to be purely imaginary 2×2 matrices, e.g.

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}.$$

The Dirac operator $i\rho^\alpha \partial_\alpha$ is then real, and it makes sense to demand that the components of

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$$

are real (Majorana spinor). Note that the ψ^μ transform in the vector representation of spacetime group $SO(D-1,1)$, but are fermions on worldsheet.

[cf. Green-Schwarz formulation]

Furthermore the above action has worldsheet supersymmetry, i.e it is invariant under infinitesimal

$$\delta X^\mu = \bar{\varepsilon}\,\psi^\mu$$

$$\delta \psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \varepsilon$$

where ε is a constant anticommuting spinor.

On the other hand, spacetime fermions (or even spacetime supersymmetry) have not yet become apparent.

The fermionic equations derived from this action is simply Dirac equation, which becomes in $\psi^\mu_\pm$ basis

$$\left(\frac{\partial}{\partial \sigma} \pm \frac{\partial}{\partial \tau} \right) \psi^\mu_\mp = 0 .$$

As before these equations of motion (together with $\Box X^\mu = 0$) must be supplemented by the super Virasoro constraints (that can be obtained as the gauge conditions starting from a suitable gauge-invariant Lagrangian):

$$0 = T_{\pm\pm} = -\frac{1}{\alpha'}\partial_\pm X^\mu \partial_\pm X_\mu - \frac{i}{2}\psi^\mu_\pm \partial_\pm \psi_{\pm\mu}$$

$$0 = J_\pm = \sqrt{\frac{2}{\alpha'}}\,\psi^\mu_\pm \partial_\pm X_\mu .$$

As before, we also need to discuss the boundary conditions for the fermions.

For closed strings, worldsheet is cylinder; the fermionic fields $\psi^\mu_\pm$ must therefore be either periodic or anti-periodic under $\sigma \mapsto \sigma + 2\pi$

Ramond

$$\psi^\mu_\pm(\sigma+2\pi,\tau) = \substack{+ \\ -}\,\psi^\mu_\pm(\sigma,\tau)$$

Neveu-Schwarz

The correct boundary conditions in the open string case, can be determined by varying the action (and requiring that super Virasoro constraints can be consistently imposed); one then finds

$$\psi_+^\mu = \overset{D}{\underset{N}{\pm}} \psi_-^\mu \quad \text{at } \sigma = 0$$

$$\psi_+^\mu = \varepsilon \left(\pm \psi_-^\mu \right) \quad \text{at } \sigma = \pi$$

$\varepsilon = +1$ Ramond

$\varepsilon = -1$ Neveu-Schwarz

In either case (by doubling trick!)

the mode expansions read

$$\psi_-^\mu(\sigma,\tau) = \sum_r b_r^\mu \, e^{-ir(\tau-\sigma)}$$

$$\psi_+^\mu(\sigma,\tau) = \sum_r \tilde{b}_r^\mu \, e^{-ir(\tau+\sigma)}$$

where

$r \in \mathbb{Z}$ Ramond

$r \in \mathbb{Z}+\frac{1}{2}$ Neveu-Schwarz

The reality of the fields requires

$$(b_r^\mu)^\dagger = b_{-r}^\mu \qquad \left((\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu \right)$$

2.1 Quantisation

As before for the bosonic case, we consider light-cone quantisation, where we solve the gauge constraints by setting $X^+ = 2\alpha' p^+ \tau$, and solve for X^- before quantisation. Now we can in addition set $\psi^+_+ = \psi^+_- = 0$ and solve for $\psi^-_\pm$. Then the independent variables are the

transverse oscillators

$$\alpha^j_n \qquad b^j_r \qquad j = 2, 3, \dots, D-1$$

together with the zero modes

$$p^+,\ x^-\ \text{and}\ x^j. \qquad [\alpha_0^j = \sqrt{2\alpha'}\, p_0^j]$$

One then imposes the canonical commutation relations for these degrees of freedom:

$$[\alpha_m^j, \alpha_n^l] = m\,\delta^{jl}\,\delta_{m,-n} \qquad [x^\mu, p^\nu] = i\eta^{\mu\nu}$$

$$\{b_r^j, b_s^l\} = \delta^{jl}\,\delta_{r,-s}.$$

States are generated by the action of creation operators from 'vacuum'

$$\alpha_n^j\,|0,p\rangle = b_r^j\,|0,p\rangle = 0 \qquad n,r>0$$

$$\alpha_0^\mu\,|0,p\rangle = \sqrt{2\alpha'}\,p^\mu\,|0,p\rangle$$

The requirement of D-dim Poincaré symmetry now fixes the mass-shell condition to be

$$\alpha' M^2 = N - \begin{cases} 1/2 & NS \\ 0 & R \end{cases}$$

where N is number operator

$$N = \sum_{\substack{n=1 \\ j}}^{\infty} \alpha_{-n}^{j} \alpha_{n}^{j} + \sum_{\substack{r>0 \\ j}} r\, b_{-r}^{j} b_{r}^{j} .$$

Furthermore, it fixes the critical dimension to be

$$D = 10 \quad \text{critical dimension}$$

Now can determine spectrum of superstring. For open string

NS sector have:

$N=0$ ground state tachyon $|0,p\rangle_{NS}$

$$M^2 = p^\mu p_\mu = -\frac{1}{2\alpha'}$$

$N=\frac{1}{2}$ massless vector field $\psi^j_{-1/2}|0,p\rangle_{NS}$

reduced from $D=10$ to $p+1$ dimensions

$N=1$ massive states $\alpha^j_{-1}|0,p\rangle_{NS}$

$\psi^i_{-1/2}\psi^j_{-1/2}|0,p\rangle_{NS}$

etc.

On the other hand, in the R-sector the only massless states arise for $N=0$. Since there are now fermionic zero modes

$$\{ b_0^i , b_0^j \} = \delta^{ij}$$

that commute with number operator, the states with $N=0$ must form a representation of above Clifford algebra of dimension

$$\dim (N=0_R) = 2^4 = 16.$$

The R-sector ground states therefore transform in the (reducible) spinor representation

$$8_s \oplus 8_c$$

of the little group $SO(8)$. In particular they therefore describe spacetime fermions.

All other states in the R-sector (that also transform in spinor reps and that are therefore also spacetime fermions) are massive.

This is not yet quite the final answer since we still have a tachyon in NS-sector. This can be rectified by GSO-projection

$$P_{GSO} = \frac{1}{2}\left(\mathbb{1} + (-1)^F\right)$$

where F is the 2d fermion number with $(-1)^F |0,p\rangle_{NS} = -|0,p\rangle_{NS}$.

This projection removes NS-tachyon, but keeps the massless vector field from the NS-sector.

Since $\{(-1)^F, b_0^j\} = 0$, $(-1)^F$ acts as a chirality operator on the R-sector ground states. Thus the GSO-projection projects onto one of the two spinor representations. After GSO-projection the open string spectrum consists of

$b^j_{-1/2}\,\|0\rangle_{NS}$	8_v	gauge boson
$\|0\rangle_R$	8_s	Majorana Weyl fermion

+ massive states.

The massless states therefore fall into the $D=10$, $N=1$ vector multiplet (dimensionally reduced to p+1 dimensions): first indication of spacetime supersymmetry.

In fact, one can show that full string spectrum (including all massive states) is spacetime supersymmetric.

[However, in NS-R formalism, spacetime supersymmetry is not manifest – cf. GS-formalism.]

For closed string theory, spectrum is essentially tensor product of left- and right-moving spectra. There are then four different sectors

$$\left.\begin{array}{l} NS \otimes NS \\ R \otimes R \end{array}\right\} \text{spacetime bosons}$$

$$\left.\begin{array}{l} NS \otimes R \\ R \otimes NS \end{array}\right\} \text{spacetime fermions}$$

Here, both left- and right-moving sectors are separately GSO-projected

$$\mathcal{P} = \frac{1}{4} \underbrace{\left(\mathbb{1} + (-1)^{F}\right)}_{R} \underbrace{\left(\mathbb{1} + (-1)^{\tilde{F}}\right)}_{L}$$

In the NS ⊗ NS sector the lowest lying massless states transform as

$$8_v \otimes 8_v = 35 \oplus 28 \oplus 1$$

graviton dilaton

Kalb-Ramond B-field

The states in the R ⊗ R sector that survive the GSO-projection depend on whether the same or the opposite spinor representation for left- and right-movers survives the projection:

IIA - theory (non-chiral), i.e opposite spinor reps survive

massless states from R⊗R sector

$$8_s \otimes 8_c = 8_v \oplus 56$$

1-form 3-form

IIB - theory (chiral), i.e. same spinor reps survive

massless states from R⊗R sector

$$8_s \otimes 8_s = 1 \oplus \underline{28} \oplus \underline{35}$$

scalar 2-form 4-form$^+$

In addition there are massive states.

The massless states in the

NS $\otimes$ R
R $\otimes$ NS sectors are

gravitini and dilatini. Together with the massless bosonic states they fall into a

D=10, N=2 supergravity multiplet.

One can also show that the full closed string spectrum is D=10, N=2 supersymmetric (but this is not manifest in NS-R formalism).

In addition to Type IIA & Type IIB there are two more supersymmetric closed string theories in $D=10$: these are the $SO(32)$ and $E_8 \times E_8$ heterotic theories, and they preserve $N=1$, $D=10$ supersymmetry.

Basic idea:

$$\underset{L}{\begin{array}{c} D=26 \\ \text{bosonic} \\ \text{string} \end{array}} \otimes \underset{R}{\begin{array}{c} D=10 \\ \text{superstring} \end{array}}$$

Finally, there is the D=10 open superstring with N=1 supersymmetry.

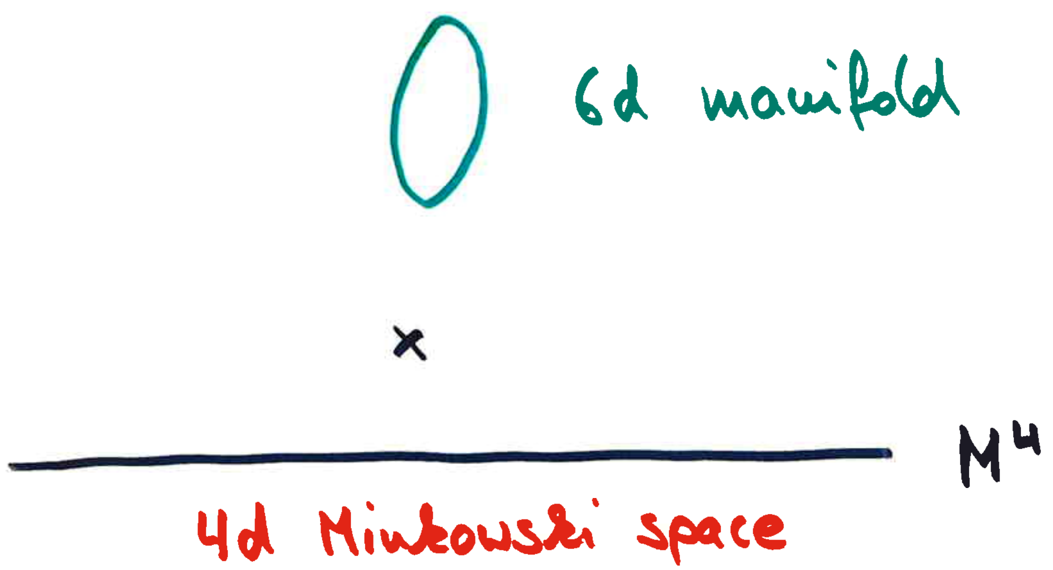
Consistency requires that it has gauge group SO(32) [≈ 32 D9-branes (+ orientifold plane)].

These 5 superstring theories are believed to be the only supersymmetric string theories in D=10.

4. Compactification & T-duality

As we have seen, the superstring is only consistent in $D = 10$ dimensions.

One way to make contact with our 4-dimensional world is to compactify six directions on a compact manifold:



In phenomenologically interesting models the internal manifold is usually a (generalised) Calabi-Yau manifold, but many of the important ideas can also be explained for the case of a torus. To simplify notation we shall furthermore only consider the case of a one-dimensional torus, i.e. of a circle S^1 with radius R.

We denote by $X(\tau,\sigma)$ the bosonic field that describes the position of the string worldsheet along this S^1.

The equations of motion still imply that

$$X(\sigma,\tau) = X_R(\tau-\sigma) + X_L(\tau+\sigma)$$

with

$$X_R = \frac{1}{2} x_R + \alpha' p_R (\tau-\sigma) + \frac{i}{2}\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n e^{-in(\tau-\sigma)}$$

$$X_L = \frac{1}{2} x_L + \alpha' p_L (\tau+\sigma) + \frac{i}{2}\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-in(\tau+\sigma)}$$

but now $p_R \neq p_L$ in general. Indeed

$$X(\sigma+2\pi,\tau) = X(\sigma,\tau) + 2\pi\alpha'(p_L-p_R)$$

but since $X \simeq X + 2\pi R$ the correct boundary condition is now

$$\alpha'(p_L - p_R) = mR$$

winding number $m \in \mathbb{Z}$

In addition there is the usual quantisation of conventional momentum

$$(p_L + p_R) = \frac{n}{R} \qquad n \in \mathbb{Z}$$

Thus

$$p_L = \frac{1}{2}\left(\frac{n}{R} + \frac{mR}{\alpha'}\right)$$

$$p_R = \frac{1}{2}\left(\frac{n}{R} - \frac{mR}{\alpha'}\right)$$

The space of states is then generated by the action of the creation operators $\alpha_n, \tilde{\alpha}_n$ with $n, \tilde{n} < 0$ on the 'vacuum'

$$|(p_L, p_R)\rangle .$$

The mass-shell condition is now

$$\frac{\alpha'}{2}\left(p_L^2 + p_R^2\right) = (N-1) + (\tilde{N}-1) - \frac{\alpha'}{4}\hat{p}^2$$

Given the above expression for (p_L, p_R)

$$\left(p_L^2 + p_R^2\right) = \frac{1}{2}\left[\left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2\right]$$

Kaluza-Klein mass

winding (string effect!)

This formula has a remarkable symmetry (T-duality): the full spectrum is invariant under replacing

$$R \longmapsto \frac{\alpha'}{R}$$

One can show that this is actually a symmetry of the full string theory, i.e.

$$\boxed{\begin{array}{c}\text{string theory}\\ \text{on } S^1 \text{ with}\\ \text{radius } R\end{array}} = \boxed{\begin{array}{c}\text{string theory}\\ \text{on } S^1 \text{ with}\\ \text{radius } \alpha'/R\end{array}}$$

In the supersymmetric case this duality transformation also exchanges $\mathrm{IIA} \leftrightarrow \mathrm{IIB}$.

T-duality is actually a good toy model for mirror symmetry

$$\underbrace{\mathrm{IIA} \text{ on } CY \longleftrightarrow \mathrm{IIB} \text{ on } \widetilde{CY}}_{\text{mirror Calabi-Yaus}}$$

[In fact, to a very good approximation

mirror symmetry = T T T]