Lattice Gauge Theory
A Short Primer

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Abstract

In this contribution we give an introduction to the foundations and methods of lattice gauge theory. Starting with a brief discussion of the quantum mechanical path integral, we develop the main ingredients of lattice field theory: functional integrals, Euclidean field theory and the space-time discretization of scalar, fermion and gauge fields. Some of the methods used in calculations are reviewed and illustrated by a collection of typical results.

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1 Introduction

1.1 Why this Article?

These lectures given at the PSI summer school 2000 in Zuoz give an overview of the basic ideas and results of lattice gauge theory for non-experts, with a stress on lattice quantum chromodynamics. This is neither a review of the status nor a survey of recent results in lattice gauge theory. Some typical results are presented for illustrative purposes only. For reviews on recent developments see [1] or the proceedings of the annual LATTICE conferences. To those who ask for a more detailed and in some places more rigorous look at this subject we recommend the literature in [2].

The article is organized as follows. We start with a motivation of non-perturbative treatments of gauge theories (especially QCD). The second chapter is an introduction to Euclidean field theory with bosons. We discuss functional integrals with imaginary time coordinates and space time discretization on a lattice at some length. Next, we sketch how to implement gauge fields on a space-time lattice, before we discuss methods used to evaluate functional integrals. In order to perform calculations in Quantum Chromodynamics (QCD), a lattice treatment of fermions is necessary, and in the fifth chapter we describe ways to put fermions on a lattice. We close with an overview of physical problems that can be addressed. Some interesting results are shown, the problem of how to approach the continuum limit is discussed and error sources are listed.

1.2 Why the Lattice?

To answer this question it is instructive to focus on a prominent example for gauge theories on a space-time lattice: QCD. Therefore let us briefly highlight the major properties of this theory: QCD originated historically as a development of the quark model. In 1961 Gell-Mann and Ne'eman established a classification scheme for the hadrons known in those days, using representations of what we call $SU(3)_{\text{flavour}}$ today. This model has one remarkable feature: the fundamental representations (typical notation is 3 and $\bar{3}$) did not appear to be realized in nature. Besides other hints this led Gell-Mann and Zweig to introduce hadrons as composite objects, with particles in the fundamental representations (the quarks and antiquarks) as their constituents. Now the quark model allows two different combination of quarks and antiquarks to exist:

$$\text{Baryons:} \qquad 3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1 \tag{1}$$

$$\text{Mesons:} \qquad 3 \otimes \bar{3} = 8 \oplus 1. \tag{2}$$

Instead of going into the details of this kinematical quark model we ask for the dynamics that allows these particular combinations: why are quarks confined in baryons and mesons, what is the mechanism that forbids the particles in the fundamental representations to be free?

The Standard Model contains QCD as a theory for quark dynamics. This gauge theory with $\mathrm{SU(3)_{colour}}$ as a gauge group introduces colour as a degree-of-freedom for the quarks (again, the quarks appear in the fundamental representation) and eight mediating vector bosons (the gluons) in the adjoint representation (they always carry two colour degrees-of-freedom and can be expanded in a basis of the eight Gell-Mann matrices).

$$\text{quark fields: } q_i^f(x), \quad i = 1,2,3, \ f = 1,\ldots,N_f \tag{3}$$

$$\text{gluon fields: } A_\mu^a(x), \quad a = 1,\ldots,8. \tag{4}$$

Here i and a are colour indices according to the relevant representation of $\mathrm{SU(3)_{colour}}$, while f labels the quark flavours u,d,s,...,N_f, and μ is the Lorentz index. The analogy to Quantum Electrodynamics (QED) (colour $\leftrightarrow$ charge and gluons $\leftrightarrow$ photons) is violated by the existence of three different colours in QCD in contrast to only one charge in QED, which reflects the different nature of the gauge groups $\mathrm{SU(3)_{colour}}$ and $\mathrm{U(1)_{hypercharge}}$. While $\mathrm{U(1)_{hypercharge}}$ is Abelian, $\mathrm{SU(3)_{colour}}$ is not. As a consequence QCD exhibits self-interaction vertices, see Fig. 1, in addition to the QED-like quark-gluon interactions shown in Fig. 2.

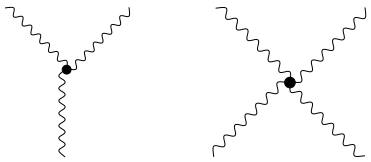

Figure 1: Gluon self–interactions

An important property of QCD is *asymptotic freedom*. The interaction strength depends on the typical energy scale Q^2 involved in the interaction process under consideration. The running of the coupling constant can be determined using renormalization group equations, with the result:

$$g_{QCD}^2(Q^2) = \frac{1}{\beta_0 \log(Q^2/\Lambda^2)} + \cdots \tag{5}$$

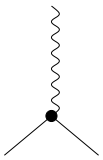

Figure 2: Quark–gluon interaction

with $\Lambda \approx 1$ GeV and

$$\beta_0 = \frac{33 - 2N_f}{48\pi^2} \geq 0, \tag{6}$$

in contrast to QED, where $\beta_0 \leq 0$.

The coupling decreases with energy in QCD. As a consequence, perturbation theory, which amounts to expansions in powers of the coupling, is phenomenologically successful for large energies, i.e. $Q^2 \gg \Lambda^2$. The predictive power of QCD at high energies will be treated in Stirling's lectures [3].

The question why quarks are confined is a question to the low energy regime of QCD and appears to require techniques beyond perturbation theory. There are other interesting low energy questions:

- *prediction of the hadron spectrum*: dynamical mass generation
- *hadron properties*: wave-functions, matrix-elements, ...
- *hadronization* in deep inelastic scattering (jet formation, ...)
- *chiral symmetry breaking* (expectation values for quark condensates, low energy constants of effective Lagrangians, ...)
- *energy scale dependence of the coupling at low* Q^2.

These are examples of non-perturbative problems in QCD.

Not only QCD, but also other components of the Standard Model and moreover theories of physics beyond the Standard Model supply us with non-perturbative problems:

- *Higgs-Yukawa models*: parameters of the Standard Model (predictions for the Higgs mass, CKM - Matrix, ...)
- *QED* (new phases, ...)

- *quantum gravity and SUSY.*

An important step to answer such questions has been made by K. Wilson in 1974 [4]. He introduced a formulation of QCD on a space-time lattice, which allows the application of various non-perturbative techniques. In the following chapters we shall explain this discretization in detail. It leads to mathematically well-defined problems, which are (at least in principle) solvable. What can be achieved in practice will be discussed below. It should also be pointed out that the introduction of a space-time lattice can be taken as a starting point for a mathematically clean approach to quantum field theory, so-called constructive quantum field theory.

2 Quantum Fields on a Lattice

This chapter is an introduction to the main concepts of lattice gauge theory: quantum field theory in its path integral formulation, Wick rotation to imaginary time coordinates leading to Euclidean field theory, and the discretization of space-time in form of a lattice. We shall illustrate these concepts with a scalar field theory.

2.1 The Quantum Mechanical Path Integral

To illuminate this crucial construction let us first consider the case of quantum mechanics of a particle in one space dimension. Let the Hamiltonian be

$$H = \frac{p^2}{2m} + V(x) \equiv H_0 + V. \tag{7}$$

The quantum mechanical transition amplitude is

$$\langle x', t'|x, t\rangle = \langle x'|e^{-iH(t'-t)}|x\rangle. \tag{8}$$

Inserting a complete set of (improper) coordinate eigenstates,

$$1 = \int dx_1 \, |x_1\rangle\langle x_1|, \tag{9}$$

into the matrix element, taking $T = (t' - t)$ and $\Delta t = (t_1 - t)$, we obtain

$$\langle x', t'|x, t\rangle = \int dx_1 \, \langle x'|e^{-iH(T-\Delta t)}|x_1\rangle\langle x_1|e^{-iH\Delta t}|x\rangle. \tag{10}$$

Dividing T into n equal parts, $T = n\Delta t$, as shown in Fig. 3, and inserting $(n-1)$ complete sets in this way, one gets

$$\langle x', t'|x, t\rangle = \int dx_1 \ldots dx_{n-1}\ \langle x'|e^{-iH\Delta t}|x_{n-1}\rangle\langle x_{n-1}|e^{-iH\Delta t}|x_{n-2}\rangle \ldots \langle x_1|e^{-iH\Delta t}|x\rangle. \quad (11)$$

In the following we set $x \equiv x_0$ and $x' \equiv x_n$.

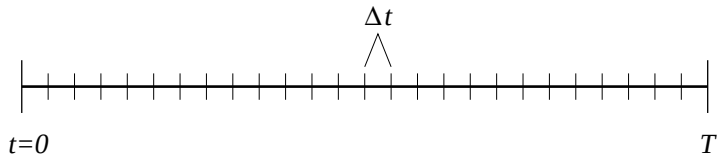

Figure 3: Discretized time interval

For large n, when Δt becomes small, we can rewrite the matrix elements by using only the first term of the Baker-Campbell-Hausdorff-formula as a good approximation to the exponential:

$$\langle x_{k+1}|e^{-iH\Delta t}|x_k\rangle \approx \langle x_{k+1}|e^{-iH_0\Delta t}e^{-iV\Delta t}|x_k\rangle\ = \langle x_{k+1}|e^{-iH_0\Delta t}|x_k\rangle\, e^{-iV(x_k)\Delta t}, \quad (12)$$

using the fact that V only depends on the space coordinates. The remaining matrix element can be calculated by means of Fourier transform with the result

$$\langle x_{k+1}|e^{-iH\Delta t}|x_k\rangle \approx \sqrt{\frac{m}{2\pi i\Delta t}}\ \exp i\Delta t\left\{\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right\}. \quad (13)$$

Doing so for every matrix element, the amplitude turns into

$$\langle x'|e^{-iHT}|x\rangle = \int \frac{dx_1 \ldots dx_{n-1}}{(\frac{2\pi i\Delta t}{m})^{n/2}} \exp\, i\sum_{k=0}^{n-1}\Delta t\left\{\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right\}. \quad (14)$$

What is this good for? In the limit $n \to \infty$ we observe that the exponent becomes the classical action

$$\sum_{k=0}^{n-1}\Delta t\left\{\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right\}$$
$$\longrightarrow \int_0^T dt\left\{\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right\} = \int_0^T dt\, L(x,\dot{x}) \equiv S \quad (15)$$

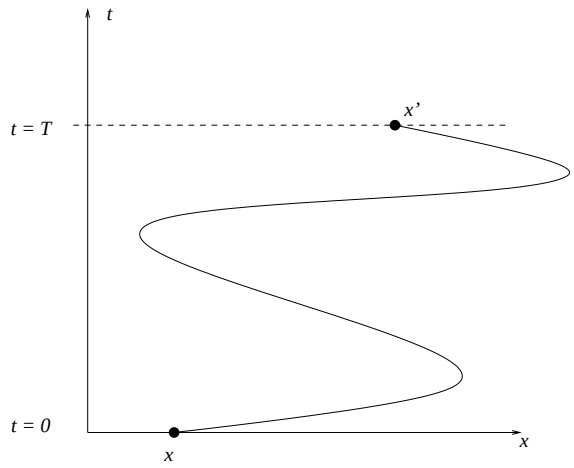

Figure 4: Path of a particle

for a path $x(t)$ from x to x' with $x_k = x(k\Delta t)$, see Fig. 4.

Second, notice that the integrations over the x_k can be interpreted as an integration of the system over all possible paths $x(t)$. Therefore we introduce the notation

$$\left(\frac{m}{2\pi i\Delta t}\right)^{n/2} dx_1 \ldots dx_{n-1} \longrightarrow \text{const.} \prod_t dx(t) \equiv \mathcal{D}x \tag{16}$$

and arrive at the path integral representation of the quantum mechanical amplitude:

$$\langle x'|e^{-iHT}|x\rangle = \int \mathcal{D}x \; e^{iS}. \tag{17}$$

For a particle in 3 dimensional space we generalize to paths $x_i(t)$, where i=1,2,3, and

$$\mathcal{D}x = \prod_t \prod_i dx_i(t). \tag{18}$$

Perhaps this is the most intuitive picture of the quantum mechanical transition amplitude. It can be written as an integral over contributions from all possible paths from the starting point to the final point. Each path is weighted by the classical action evaluated along this path.

2.2 Quantum Field Theory with Functional Integrals

Now we are going to translate to field theory this representation of quantum mechanics in terms of path integrals. We consider a scalar field $\phi(x)$, where

$x = (\vec{x}, t)$ labels space-time coordinates, and the time evolution of $\phi(\vec{x}, t)$ is given by

$$\phi(\vec{x}, t) = e^{iHt}\, \phi(\vec{x}, t = 0)\, e^{-iHt}. \tag{19}$$

The objects of interest in field theory are vacuum expectation values of (time ordered) products of field operators, the Greens functions:

$$\langle 0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle, \qquad t_1 > t_2 > \ldots > t_n. \tag{20}$$

Prominent examples are propagators

$$\langle 0|\phi(x)\phi(y)|0\rangle. \tag{21}$$

The Greens functions essentially contain all physical information. In particular, S-matrix elements are related to Greens functions, e.g. the 2-particle scattering elements can be obtained from

$$\langle 0|\phi(x_1)\ldots\phi(x_4)|0\rangle. \tag{22}$$

Instead of discussing the functional integral representation for quantum field theory from the beginning, we shall restrict ourselves to translating the quantum mechanical concepts to field theory by means of analogy. To this end we would like to translate the basic variables $x_i(t)$ into fields $\phi(\vec{x}, t)$. The rules for the translation are then

$$\begin{aligned}
x_i(t) &\longleftrightarrow \phi(\vec{x}, t) \\
i &\longleftrightarrow \vec{x} \\
\prod_{t,i} dx_i(t) &\longleftrightarrow \prod_{t,\vec{x}} d\phi(\vec{x}, t) \equiv \mathcal{D}\phi \\
S = \int dt\ L &\longleftrightarrow S = \int dt\, d^3x\ \mathcal{L},
\end{aligned}$$

where S is the classical action.

For scalar field theory we might consider the following Lagrangian density:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left((\dot{\phi}(x))^2 - (\nabla\phi(x))^2\right) - \frac{m_0^2}{2}\phi(x)^2 - \frac{g_0}{4!}\phi(x)^4 \\
&= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m_0^2}{2}\phi(x)^2 - \frac{g_0}{4!}\phi(x)^4.
\end{aligned} \tag{23}$$

The mass m_0 and coupling constant g_0 bear a subscript 0, since they are bare, unrenormalized parameters. This theory plays a role in the context of Higgs-Yukawa models, where $\phi(x)$ is the Higgs field.

In analogy to the quantum mechanical path integral we now write down a representation of the Greens functions in terms of what one calls *functional integrals*:

$$\langle 0|\phi(x_1)\phi(x_2)\dots\phi(x_n)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\dots\phi(x_n)\, e^{iS} \qquad (24)$$

with

$$Z = \int \mathcal{D}\phi\ e^{iS}. \qquad (25)$$

These expressions involve integrals over all classical field configurations.

As mentioned before, we do not attempt any derivation of functional integrals but just want to motivate their form by analogy. Furthermore, in the case of quantum mechanics we considered the transition amplitude, whereas now we have written the formula for Greens functions, which is a bit different.

The formulae for functional integrals give rise to some questions. First of all, how does the projection onto the groundstate $|0\rangle$ arise? Secondly, these integrals contain oscillating integrands, due to the imaginary exponents; what about their convergence? Moreover, is there a way to evaluate them numerically?

In the following we shall discuss, how the introduction of imaginary times helps in answering these questions.

2.3 Euclidean Field Theory

Let us return to quantum mechanics for a moment. Here we can also introduce Greens functions, e.g.

$$G(t_1,t_2) = \langle 0|X(t_1)X(t_2)|0\rangle, \qquad t_1 > t_2. \qquad (26)$$

We are now going to demonstrate that these Greens functions are related to quantum mechanical amplitudes at imaginary times by analytic continuation. Consider the matrix element

$$\langle x',t'|X(t_1)X(t_2)|x,t\rangle = \langle x'|e^{-iH(t'-t_1)}Xe^{-iH(t_1-t_2)}Xe^{-iH(t_2-t)}|x\rangle \qquad (27)$$

for $t' > t_1 > t_2 > t$. Now let us choose all times to be purely imaginary

$$t = -i\tau, \qquad (28)$$

again ordered, $\tau' > \tau_1 > \tau_2 > \tau$. This yields the expression

$$\langle x'|e^{-H(\tau'-\tau_1)}Xe^{-H(\tau_1-\tau_2)}Xe^{-H(\tau_2-\tau)}|x\rangle. \tag{29}$$

Inserting a complete set of energy eigenstates, the expansion of the time evolution operator in imaginary times is

$$e^{-H\tau} = \sum_{n=0}^{\infty} e^{-E_n\tau}|n\rangle\langle n| = |0\rangle\langle 0| + e^{-E_1\tau}|1\rangle\langle 1| + \ldots, \tag{30}$$

where the ground state energy has been normalized to $E_0 = 0$. For large τ it reduces to the projector onto the groundstate. Consequently, in the limit $\tau' \to \infty$ and $\tau \to -\infty$ our matrix element becomes

$$\langle x'|0\rangle\langle 0|Xe^{-H(\tau_1-\tau_2)}X|0\rangle\langle 0|x\rangle, \tag{31}$$

and similarly

$$\langle x'|e^{-H(\tau'-\tau)}|x\rangle \longrightarrow \langle x'|0\rangle\langle 0|x\rangle. \tag{32}$$

Therefore the Greens function at imaginary times,

$$G_E(\tau_1,\tau_2) = \langle 0|Xe^{-H(\tau_1-\tau_2)}X|0\rangle, \tag{33}$$

can be expressed as

$$G_E(\tau_1,\tau_2) = \lim_{\substack{\tau'\to\infty \\ \tau\to-\infty}} \frac{\langle x'|e^{-H(\tau'-\tau_1)}Xe^{-H(\tau_1-\tau_2)}Xe^{-H(\tau_2-\tau)}|x\rangle}{\langle x'|e^{-H(\tau'-\tau)}|x\rangle}. \tag{34}$$

Now we can represent the denominator as well as the numerator by path integrals as seen before. The difference is that for imaginary times we have to use

$$\langle x|e^{-H\Delta\tau}|y\rangle \approx \sqrt{\frac{m}{2\pi\Delta\tau}}\ \exp -\Delta\tau\left\{\frac{m}{2}\left(\frac{x-y}{\Delta\tau}\right)^2 + V(x)\right\}. \tag{35}$$

This leads to the path integral representation

$$G_E(\tau_1,\tau_2) = \frac{1}{Z}\int \mathcal{D}x\ x(\tau_1)x(\tau_2)\, e^{-S_E}, \tag{36}$$

where

$$Z = \int \mathcal{D}x\ e^{-S_E} \tag{37}$$

and

$$S_E = \int d\tau \left\{\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x(\tau))\right\}. \tag{38}$$

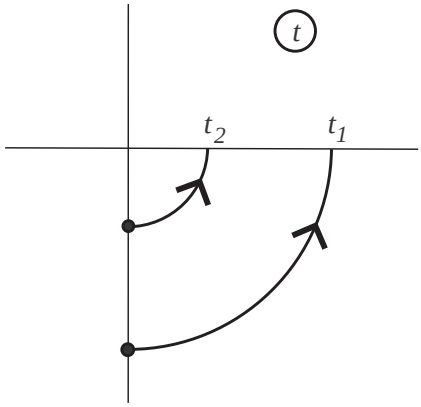

Figure 5: Wick rotation from imaginary to real time coordinates

The Greens function at real times, which we were interested in originally, can be obtained from G_E by means of analytical continuation, $G(t_1, t_2) = G_E(it_1, it_2)$. The analytical continuation has to be done in such a way that all time arguments are rotated simultaneously counter-clockwise in the complex t-plane. This is the so-called Wick rotation, illustrated in Fig. 5.

Now we turn to field theory again. The Greens functions

$$G(x_1, \ldots, x_n) = \langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle, \tag{39}$$

continued to imaginary times, $t = -i\tau$, are the so-called Schwinger functions

$$G_E((\vec{x}_1, \tau_1), \ldots, (\vec{x}_n, \tau_n)) = G((\vec{x}_1, -i\tau_1), \ldots, (\vec{x}_n, -i\tau_n)). \tag{40}$$

In analogy to the quantum mechanical case their functional integral representation reads

$$G_E(x_1, \ldots, x_n) = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(x_1)\ldots\phi(x_n)\, e^{-S_E} \tag{41}$$

with

$$Z = \int \mathcal{D}\phi\ e^{-S_E} \tag{42}$$

and

$$\begin{aligned} S_E &= \int d^3x d\tau \left\{ \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{g_0}{4!}\phi^4 \right\} \\ &= \int d^4x \left\{ \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{g_0}{4!}\phi^4 \right\}. \end{aligned} \tag{43}$$

As can also be seen from the kinetic part contained in S_E, the metric of Minkowski space

$$-ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \tag{44}$$

has changed into

$$d\tau^2 + dx_1^2 + dx_2^2 + dx_3^2, \tag{45}$$

which is the metric of a Euclidean space. Therefore one speaks of *Euclidean Greens functions* G_E and of *Euclidean functional integrals*. They are taken as starting point for non-perturbative investigations of field theories and for constructive studies.

As S_E is real, the integrals of interest are now real and no unpleasant oscillations occur. Moreover, since S_E is bounded from below, the factor $\exp(-S_E)$ in the integrand is bounded. Strongly fluctuating fields have a large Euclidean action S_E and are thus suppressed by the factor $\exp(-S_E)$. (Strictly speaking, this statement does not make sense in field theory unless renormalization is taken into account.) This makes Euclidean functional integrals so attractive compared to their Minkowskian counterparts.

To illustrate the coordinate transformation to imaginary time, there is a little exercise. Consider the Feynman propagator and show that

$$\Delta_F^E(x) = \int \frac{d^4p}{(2\pi)^4} \, \frac{e^{ipx}}{p^2 + m_0^2}, \tag{46}$$

(where px is to be understood as a Euclidean scalar product), is obtained by correct Wick rotation. To be more precise,

$$\Delta_F(\vec{x}, t) = \lim_{\phi \to \pi/2} \Delta_F^E(\vec{x}, te^{i\phi}), \tag{47}$$

with Δ_F the Feynman propagator in Minkowski-space

$$\Delta_F(\vec{x}, t) = i \int \frac{d^4p}{(2\pi)^4} \, \frac{e^{-ip*x}}{p^2 - m_0^2 + i\epsilon}, \tag{48}$$

where all scalar products in the last expression are defined with Minkowski metric. An important feature of the Wick-rotated propagator is the absence of singularities on the p^4-axis in Euclidean space, see Fig. 6.

One might think that in the Euclidean domain everything is unphysical and there is no possibility to get physical results directly from the Euclidean Greens functions. But this is not the case. For example, the spectrum of the theory can be obtained in the following way. Let us consider a vacuum expectation value of the form

$$\langle 0 | A_1 e^{-H\tau} A_2 | 0 \rangle, \tag{49}$$

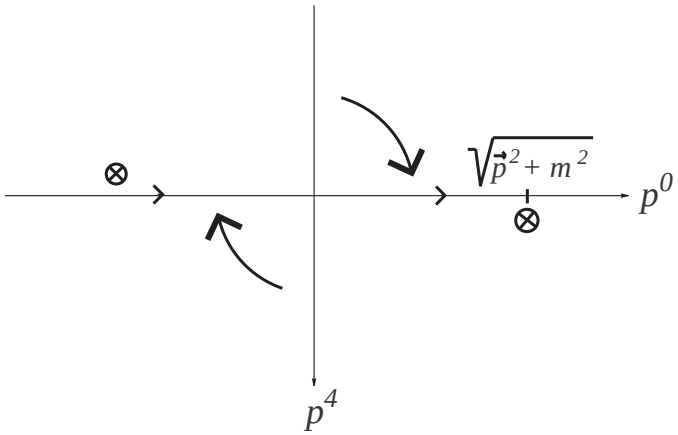

Figure 6: Wick rotation in momentum space and the position of the propagator poles

where the A_i's are formed out of the field ϕ, e.g. $A = \phi(\vec{x}, 0)$ or $A = \int d^3x\ \phi(\vec{x}, 0)$. Now, with the familiar insertion of a complete set of energy eigenstates, we have

$$\langle 0|A_1 e^{-H\tau} A_2|0\rangle = \sum_n \langle 0|A_1|n\rangle e^{-E_n\tau}\langle n|A_2|0\rangle. \tag{50}$$

In case of a continuous spectrum the sum is to be read as an integral. On the other hand, representing the expectation value as a functional integral leads to

$$\frac{1}{Z}\int \mathcal{D}\phi\ e^{-S_E} A_1(\tau)A_2(0) = \sum_n \langle 0|A_1|n\rangle\langle n|A_2|0\rangle e^{-E_n\tau}. \tag{51}$$

This is similar to the ground state projection at the beginning of this chapter. For large τ the lowest energy eigenstates will dominate the sum and we can thus obtain the low-lying spectrum from the asymptotic behaviour of this expectation value. By choosing A_1, A_2 suitably, e.g. for

$$A \equiv A_1 = A_2 = \int d^3x\ \phi(\vec{x}, 0), \tag{52}$$

such that $\langle 0|A|1\rangle \neq 0$ for a one-particle state $|1\rangle$ with zero momentum $\vec{p} = 0$ and mass m_1, we will get

$$\frac{1}{Z}\int \mathcal{D}\phi\ e^{-S_E} A(\tau)A(0) = |\langle 0|A|1\rangle|^2 e^{-m_1\tau} + \ldots, \tag{53}$$

which means that we can extract the mass of the particle.

From now on we shall remain in Euclidean space and suppress the subscript E, so that $S \equiv S_E$ means the Euclidean action.

2.4 Lattice Discretization

One central question still remains: does the infinite dimensional integration over all classical field configurations, i.e.

$$\mathcal{D}\phi = \prod_x d\phi(x), \tag{54}$$

make sense at all? How is it defined?

Remember the way we derived the path integral representation of quantum mechanics. It was obtained as a limit of a discretization in time τ. As in field theory the fields depend on the four Euclidean coordinates instead of a single time coordinate, we may now introduce a discretized space-time in form of a lattice, for example a hypercubic lattice, specified by

$$x_\mu = a n_\mu, \qquad n_\mu \in \mathbf{Z}, \tag{55}$$

see Fig. 7.

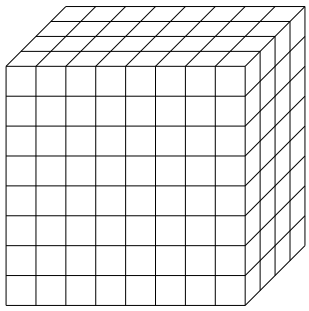

Figure 7: 3–dimensional lattice

The quantity a is called the lattice spacing for obvious reasons. The scalar field

$$\phi(x), \qquad x \in \text{lattice}, \tag{56}$$

is now defined on the lattice points only. Partial derivatives are replaced by finite differences,

$$\partial_\mu \phi \longrightarrow \Delta_\mu \phi(x) \equiv \frac{1}{a}(\phi(x + a\hat{\mu}) - \phi(x)), \tag{57}$$

and space-time integrals by sums:

$$\int d^4x \quad \longrightarrow \sum_x a^4 \ . \tag{58}$$

The action of our discretized ϕ^4-theory, Eq. (43), can be written as

$$S = \sum_x a^4 \left\{ \frac{1}{2} \sum_{\mu=1}^{4} (\Delta_\mu \phi(x))^2 + \frac{m_0^2}{2} \phi(x)^2 + \frac{g_0}{4!} \phi(x)^4 \right\}. \tag{59}$$

In the functional integrals the measure

$$\mathcal{D}\phi = \prod_x d\phi(x) \tag{60}$$

involves the lattice points x only. So we have a discrete set of variables to integrate. If the lattice is taken to be finite, we just have finite dimensional integrals.

Discretization of space-time using lattices has one very important consequence. Due to a non-zero lattice spacing a cutoff in momentum space arises. The cutoff can be observed by having a look at the Fourier transformed field

$$\tilde{\phi}(p) = \sum_x a^4 \; e^{-ipx} \; \phi(x). \tag{61}$$

The Fourier transformed functions are periodic in momentum-space, so that we can identify

$$p_\mu \cong p_\mu + \frac{2\pi}{a} \tag{62}$$

and restrict the momenta to the so-called Brillouin zone

$$-\frac{\pi}{a} \; < \; p_\mu \; \leq \frac{\pi}{a}. \tag{63}$$

The inverse Fourier transformation, for example, is given by

$$\phi(x) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \; e^{ipx} \; \tilde{\phi}(p). \tag{64}$$

We recognize an ultraviolet cutoff

$$|p_\mu| \leq \frac{\pi}{a}. \tag{65}$$

Therefore field theories on a lattice are regularized in a natural way.

In order to begin in a well-defined way one would start with a finite lattice. Let us assume a hypercubic lattice with length $L_1 = L_2 = L_3 = L$ in every spatial direction and length $L_4 = T$ in Euclidean time,

$$x_\mu = a n_\mu, \qquad n_\mu = 0, 1, 2, \ldots, L_\mu - 1, \tag{66}$$

with finite volume $V = L^3T$. In a finite volume one has to specify boundary conditions. A popular choice are periodic boundary conditions

$$\phi(x) = \phi(x + aL_\mu\,\hat{\mu}), \tag{67}$$

where $\hat{\mu}$ is the unit vector in the μ-direction. They imply that the momenta are also discretized,

$$p_\mu = \frac{2\pi}{a}\,\frac{l_\mu}{L_\mu} \qquad \text{with } l_\mu = 0, 1, 2, \ldots, L_\mu - 1, \tag{68}$$

and therefore momentum-space integration is replaced by finite sums

$$\int \frac{d^4p}{(2\pi)^4} \;\longrightarrow\; \frac{1}{a^4L^3T}\sum_{l_\mu}. \tag{69}$$

Now, all functional integrals have turned into regularized and finite expressions.

Of course, one would like to recover physics in a continuous and infinite space-time eventually. The task is therefore to take the infinite volume limit,

$$L, T \longrightarrow \infty, \tag{70}$$

which is the easier part in general, and to take the the continuum limit,

$$a \longrightarrow 0. \tag{71}$$

Constructing the continuum limit of a lattice field theory is usually highly nontrivial and most effort is often spent here.

The formulation of Euclidean quantum field theory on a lattice bears a useful analogy to statistical mechanics. Functional integrals have the form of partition functions and we can set up the following correspondence:

Euclidean field theory	Statistical Mechanics
generating functional $\int \mathcal{D}\phi\; e^{-S}$	partition function $\sum e^{-\beta\mathcal{H}}$
action S	Hamilton function $\beta\mathcal{H}$
mass m $G \sim e^{-mt}$	inverse correlation length $1/\xi$ $G \sim e^{-\frac{x}{\xi}}$

This formal analogy allows to use well established methods of statistical mechanics in field theory and vice versa. Even the terminology of both fields is often identical. To mention some examples, in field theory one employs high-temperature expansions and mean field approximations, and in statistical mechanics one applies the renormalization group.

3 Lattice Gauge Theory

In this section gauge fields are implemented on a space-time lattice. After introducing the concept of parallel transporters we define the variables of lattice gauge theory and consider some aspects of pure gauge theory, including static confinement and the glueball spectrum.

3.1 Parallel Transporters

Let us start with a brief reminder of gauge transformations in continuum field theory. For an N-component complex scalar field $\phi(x) = (\phi^i(x)), \quad i = 1, \ldots, N$, gauge transformations are defined as

$$\phi(x) \longrightarrow \Lambda(x)\phi(x), \qquad \text{with } \Lambda(x) \in \mathrm{SU}(N), \tag{72}$$

where we consider the case of gauge group $\mathrm{SU}(N)$.

In order to restore the invariance of the Lagrangian under these transformations, one introduces a covariant derivative

$$D_\mu\phi(x) = (\partial_\mu - ig_0 A^a_\mu(x) T_a)\,\phi(x), \tag{73}$$

where the T^a's are generators of the gauge group and $A^a_\mu(x)$ is the gauge field. For $\mathrm{SU}(N)$ there are (N^2-1) generators T_a, $a = 1, 2, 3, \ldots, N^2-1$, and they satisfy

$$[T_a, T_b] = i f_{abc} T_c \tag{74}$$

with the structure constants f_{abc} of $\mathrm{SU}(N)$. For SU(2) the three generators are given by the Pauli matrices

$$T_a = \frac{\sigma_a}{2}, \qquad a = 1, 2, 3, \tag{75}$$

and for SU(3) they are the Gell-Mann matrices

$$T_a = \frac{\lambda_a}{2}, \qquad a = 1, \ldots, 8. \tag{76}$$

It is now an easy exercise to show that the covariant derivative of a scalar field transforms covariantly under the gauge transformation:

$$D_\mu\phi(x) \longrightarrow \Lambda(x) D_\mu\phi(x), \tag{77}$$

and therefore a kinetic term $D_\mu\phi(x) \cdot D_\mu\phi(x)$ is invariant under gauge transformations.

Associated with the gauge field is its field strength

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f_{abc} A^b_\mu A^c_\nu. \tag{78}$$

With the help of it an action for the gauge field, the Yang-Mills action, can be written down as

$$S_{YM} = \frac{1}{4} \int d^4x \; F^a_{\mu\nu}(x) F^a_{\mu\nu}(x). \tag{79}$$

Now let us consider a product of fields at different points,

$$\phi(x) \cdot \phi(y) = \sum_i \overline{\phi^i(x)}\, \phi^i(y). \tag{80}$$

This is not invariant under gauge transformations. But such terms will occur, if we write down a kinetic term in lattice field theory. Therefore, we need matrices $U(x,y) \in \mathrm{SU}(N)$ which transform as $U(x,y) \longrightarrow \Lambda(x) U(x,y) \Lambda^{-1}(y)$, such that $\phi(x) \cdot U(x,y) \phi(y)$ would be invariant. There is a solution to this problem. Take a path $\mathcal{C}$ from y to x and define

$$U(x,y;\mathcal{C}) \equiv P \exp ig_0 \int_y^x A^a_\mu(z) T_a dz^\mu, \tag{81}$$

where the integral is taken along the path $\mathcal{C}$ and the symbol P indicates a path ordering of the non-commuting factors $A^a_\mu(z) T_a$ like the time ordering in Dyson's formula.

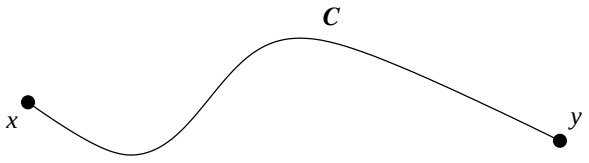

Figure 8: Path $\mathcal{C}$ between y and x

Then $U(x,y;\mathcal{C})$ transforms as desired and fulfills the goal. It is called *parallel transporter* in analogy to similar objects in differential geometry, which map vectors from one point to another along curves. The parallel transporters depend not only on the points x and y but also on the chosen curve $\mathcal{C}$. They obey the composition rule

$$U(x,y;\mathcal{C}) = U(x,u;\mathcal{C}_1) \cdot U(u,y;\mathcal{C}_2), \tag{82}$$

where the path $\mathcal{C}$ is split into two parts $\mathcal{C}_1$ and $\mathcal{C}_2$.

In the Abelian case, where the gauge group is U(1), the path ordering P is not required and we just have

$$U(x,y;\mathcal{C}) = \exp ig_0 \int_y^x A_\mu(z) dz^\mu. \tag{83}$$

3.2 Lattice Gauge Fields

The kinetic term for scalar fields on a lattice involves the product of fields at neighbouring lattice points, separated by the lattice spacing a. In order to make it gauge invariant we need the smallest parallel transporters, which exist on a lattice, namely those connecting nearest neighbour points. The corresponding paths are called *links*.

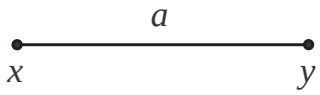

Figure 9: Link $b = \langle x, y \rangle$ between lattice points x and y

With each link

$$b = \langle x + a\hat{\mu}, x \rangle \tag{84}$$

in lattice direction $\hat{\mu}$ we associate a sort of elementary parallel transporter, the link variable

$$U(b) \equiv U(x + a\hat{\mu}, x) \equiv U_{x\mu} \in \mathrm{SU}(N). \tag{85}$$

They replace the gauge field $A^a_\mu(x)$. The link variables transform as

$$U(x, y) \longrightarrow \Lambda(x) U(x, y) \Lambda^{-1}(y) \tag{86}$$

and therefore in the action a term of the form

$$\sum_{x,\mu} \phi(x + a\hat{\mu}) \cdot U_{x\mu}\, \phi(x) \tag{87}$$

is invariant. There exist other gauge invariant expressions on the lattice. Particularly interesting is the trace of a product of link variables along a closed path,

$$\mathrm{Tr}(U(b_1) U(b_2) \ldots U(b_n)). \tag{88}$$

The most elementary one is a plaquette, as shown in Fig. 10.

The plaquette variable

$$U(p) = U_{x\mu\nu} \equiv U_{(x+a\hat{\nu})(-\nu)} U_{(x+a\hat{\mu}+a\hat{\nu})(-\mu)} U_{(x+a\hat{\mu})\nu} U_{x\mu} \tag{89}$$

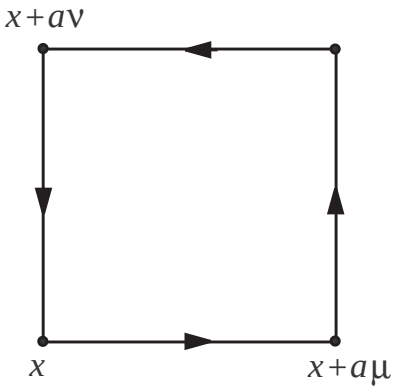

Figure 10: Plaquette p in lattice directions μ and ν

has been used by Wilson to construct a lattice Yang-Mills action. He proposed

$$S_W = -\sum_p \frac{\beta}{N}\mathrm{Re}(\mathrm{Tr}(U(p))), \tag{90}$$

which is gauge invariant and purely real by construction. The constant β is to be determined in the continuum limit requiring that the standard Yang-Mills action is recovered in that case. If we introduce gauge field variables by

$$U_{x\mu} \equiv \exp ig_0 a A^b_\mu(x) T_b \tag{91}$$

then we obtain in a naive continuum limit, where a goes to zero,

$$S_W = \frac{\beta g_0^2}{8N}\sum_x a^4 F^b_{\mu\nu}F^b_{\mu\nu} + \mathcal{O}(a^5), \tag{92}$$

and we can read off

$$\beta = \frac{2N}{g_0^2}. \tag{93}$$

For the quantum theory we have to specify how to do functional integrals. The integral over all gauge field configurations on the lattice amounts to an integral over all link variables $U(b)$. So, for the expectation value of any observable A we write

$$\langle A\rangle = \frac{1}{Z}\int\prod_b dU(b)\ A\ e^{-S_W}, \tag{94}$$

where the integration $dU(b)$ for a given link b is to be understood as the invariant integration over the group manifold (e.g. a 3-sphere for SU(2)), normalized to

$$\int dU = 1. \tag{95}$$

As a shorthand, we shall write

$$\mathcal{D}U \equiv \prod_b dU(b). \tag{96}$$

It is worth noticing here that no gauge fixing appears to be necessary. The total "volume of the gauge group" is unity. Gauge fixing is required for the purpose of perturbation theory only.

3.3 Some Observables

Already in pure gauge theory some quantities of physical interest occur and we shall consider two of them.

1. The Wilson loop is defined as the trace of a parallel transporter for a closed curve $\mathcal{C}$ of spatial length R and time extension T, see Fig. 11.

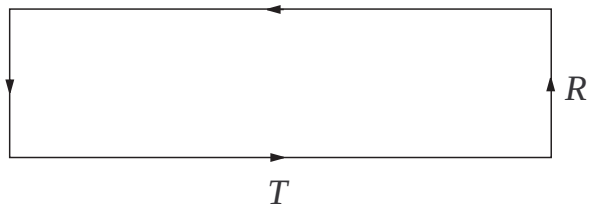

Figure 11: Wilson loop

In the limit $T \to \infty$ it can be shown that

$$\langle \mathrm{Tr}(U(\mathcal{C})) \rangle \sim C \exp -TV(R), \tag{97}$$

with $V(R)$ being the potential between static colour charges, the so-called static quark-antiquark potential.

If for large areas RT the Wilson loop goes to zero with an area law of the form

$$\langle \mathrm{Tr}(U(\mathcal{C})) \rangle \sim \exp -\alpha RT, \tag{98}$$

the potential rises linearly for large R:

$$V(R) \sim \alpha R. \tag{99}$$

Such a situation is called static quark confinement, since colour charges feel a constant attractive force at arbitrary large distances.

2. Plaquette correlations are expectation values of the product of two spatial plaquettes p_1 and p_2, separated by a time t as illustrated in Fig. 12.

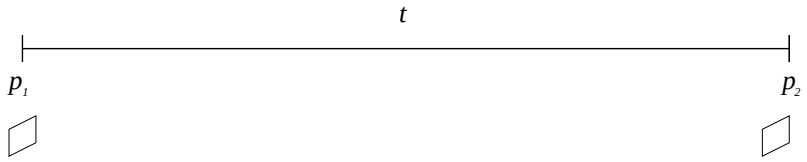

Figure 12: Plaquette–plaquette correlations

In non-abelian lattice gauge theories one finds that these correlations fall off exponentially according to

$$\langle \mathrm{Tr}(U(p_1))\mathrm{Tr}(U(p_2))\rangle_c \sim \exp -mt. \tag{100}$$

From our general discussion it follows that m is the lowest particle mass in the theory. Since there are only gluonic degrees of freedom present, the corresponding massive particle is called *glueball*.

4 Methods

In the previous section the functional integrals for gauge theories on the lattice have been defined. But it is another problem to evaluate these high-dimensional integrals. A calculation in closed form appears to be impossible in general. In this section we shall consider some of the methods used to evaluate the functional integrals approximately.

4.1 Perturbation Theory

Although lattice gauge theory offers us the possibility to study non-perturbative aspects, perturbation theory is nevertheless a highly valuable tool on the lattice, too. In particular, it is used to relate perturbatively and non-perturbatively calculated quantities.

Perturbation theory amounts to an expansion in powers of the coupling as in the continuum. The lattice provides an intrinsic UV cutoff π/a for all momenta. Apart from that one has to observe that the propagators and vertices are different from the continuum ones, owing to the form of the lattice action. In particular, gluon self interactions of all orders appear and not only as three and four gluon vertices. As mentioned before, perturbation theory on the lattice requires gauge fixing as is the case in the continuum.

4.2 Strong Coupling Expansion

We have already pointed out the analogies between Euclidean field theory and statistical mechanics. In statistical mechanics a well-established technique is the high-temperature expansion. This is an expansion in powers of

$$\beta \sim \frac{1}{g_0^2}, \tag{101}$$

which is a small quantity at large bare couplings g_0. Therefore it is the same as a strong coupling expansion. Basically the Boltzmann factor is expanded as

$$\exp \beta \frac{1}{N} \mathrm{Re}(\mathrm{Tr}(U(p))) = 1 + \beta \frac{1}{N} \mathrm{Re}(\mathrm{Tr}(U(p))) + \dots \ . \tag{102}$$

The resulting expansion can be represented diagrammatically, similar to the Feynman diagrams of perturbation theory. The diagram elements, however, are plaquettes p on the lattice. Every power of β introduces one more plaquette.

For example, the calculation of a Wilson loop in the strong coupling expansion leads to surfaces that are bounded by the loop $\mathcal{C}$, see Fig. 13.

Figure 13: Leading terms in the diagrammatic representation of the strong coupling expansion of a Wilson loop

These diagrams lead to an area law of the form

$$\langle \mathrm{Tr}(U(\mathcal{C})) \rangle \sim e^{-\alpha RT} \qquad \text{with} \quad \alpha = -\ln \beta + \dots = \ln g_0^2 + \dots . \tag{103}$$

The strong coupling expansion of plaquette correlations is associated with surfaces connecting the two plaquettes. The smallest of them is a tube of cross section 1 between the plaquettes, with total area $4t$, see Fig. 14.

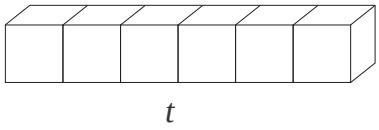

Figure 14: Leading term in the diagrammatic representation of the strong coupling expansion of a plaquette correlation

Summing these diagrams gives

$$\langle \mathrm{Tr}(U(p_1))\mathrm{Tr}(U(p_2))\rangle_c \sim e^{-mt} \qquad \text{with} \quad m = -4\ln\beta + \ldots . \tag{104}$$

This is a general result for lattice gauge theories: for small β, i.e. for large bare couplings g_0, there is static confinement, $\alpha > 0$, and dynamical mass generation, $m > 0$. These are genuine non-perturbative properties.

4.3 Other Analytic Methods

Other analytical methods are available for approximative evaluations of the functional integrals of lattice gauge theory. Instead of going into the details we just want to mention some of them:

- mean field approximation
- renormalization group
- $\frac{1}{N}$-expansion

4.4 Monte Carlo Methods

On a finite lattice the calculation of expectation values requires the evaluation of finite dimensional integrals. This immediately suggests the application of numerical methods. The first thing one would naively propose is some simple numerical quadrature. In order to understand that this approach wouldn't be all that helpful, let us consider a typical lattice as it is considered in recent calculations. With 40 lattice points in every direction we have $4 \cdot 40^4$

link variables. For gauge group SU(3) this gives 81,920,000 real variables. That should be intractable for conventional quadratures even in the future. Therefore some statistical method is required. Producing lattice gauge configurations just randomly turns out to be extremely inefficient. The crucial idea to handle this problem is the concept of *importance sampling*: for a given lattice action S quadrature points x_i are generated with a probability

$$p(x_i) \sim \exp -S(x_i). \tag{105}$$

This provides us with a large number of points in the important regions of the integral, improving the accuracy drastically. See Fig. 15 for a one-dimensional sketch.

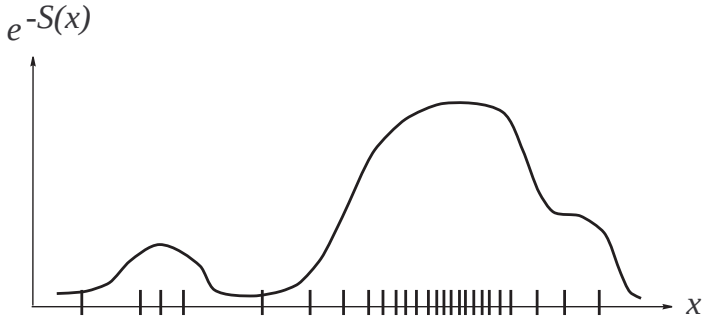

Figure 15: One-dimensional distribution of importance sampled quadrature points

In case of lattice gauge theory the quadrature points are configurations $U^{(i)} = \left\{U_{x\mu}^{(i)}\right\}$. An expectation value

$$\langle 0|A|0\rangle = \frac{1}{Z}\int \mathcal{D}U \; A(U) \; e^{-S(U)} \tag{106}$$

is numerically approximated by the average

$$\bar{A} \equiv \frac{1}{n}\sum_{i=1}^{n} A(U^{(i)}). \tag{107}$$

The Monte Carlo method consists in producing a sequence of configurations $U^{(1)} \to U^{(2)} \to U^{(3)} \to \ldots$ with the appropriate probabilities in a statistical way. This is of course done on a computer. An *update* is a step where a single link variable $U_{x\mu}$ is changed, whereas a *sweep* implies that one

goes once through the entire lattice, updating all link variables. A commonly used technique for obtaining updates is the *Metropolis algorithm.*

An important feature of this statistical way of evaluation is the existence of statistical errors. The result of such a calculation is usually presented in the form

$$\langle A \rangle = \bar{A} \pm \sigma_{\bar{A}}, \tag{108}$$

where the variance of $\bar{A}$ decreases with the number n of configurations as

$$\sigma_{\bar{A}} \sim \frac{1}{n^{1/2}}. \tag{109}$$

5 Fermions on the Lattice

Maybe the reader is already asking themself: when do they finally start discussing QCD? To approach the strong interactions, an implementation of quark fields on the lattice is necessary. We shall demonstrate the foundations, techniques and difficulties of that task reviewing the standard representation of fermions with Grassmann variables and discussing several ansaetze to discretize this continuum description. This will lead us to the question of fermion doubling on the lattice - a special problem in the chiral limit.

5.1 Grassmann Variables

First remember scalar fields in the continuum. Classical fields are just ordinary functions and satisfy

$$[\phi(x), \phi(y)] = 0, \tag{110}$$

which can be considered as the limit $\hbar \to 0$ of the quantum commutation relations.

Fermi statistics implies that fermionic quantum fields have the well-known equal-time anticommutation relations

$$\{\psi(\vec{x}, t), \psi(\vec{y}, t)\} = 0. \tag{111}$$

Motivated by this, we might introduce a classical limit in which classical fermionic fields satisfy

$$\{\psi(x), \psi(y)\} = 0 \tag{112}$$

for all x, y. Classical fermionic fields are therefore anticommuting variables, which are also called *Grassmann variables*.

We would like to point out that the argument above is just a heuristic motivation. More rigorous approaches can be found in the literature.

In general, a complex Grassmann algebra is generated by elements η_i and $\bar{\eta}_i$, which obey

$$\{\eta_i, \eta_j\} = 0 \tag{113}$$
$$\{\eta_i, \bar{\eta}_j\} = 0 \tag{114}$$
$$\{\bar{\eta}_i, \bar{\eta}_j\} = 0. \tag{115}$$

An integration of Grassmann variables can be defined by

$$\int d\eta_i \; (a + b\eta_i) = b \tag{116}$$

for arbitrary complex numbers a, b.

In fermionic field theories we have Grassmann fields, which associate Grassmann variables whith every space-time point. For example, a Dirac field has anticommuting variables $\psi_\alpha(x)$ and $\bar{\psi}_\alpha(x)$, where α=1,2,3,4 is the Dirac index. The classical Dirac field obeys

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0, \quad \text{etc.}. \tag{117}$$

In order to write down fermionic path integrals as integrals over fermionic and anti-fermionic field configurations, we write

$$\mathcal{D}\psi \; \mathcal{D}\bar{\psi} = \prod_x \prod_\alpha d\psi_\alpha(x) \, d\bar{\psi}_\alpha(x). \tag{118}$$

Then any fermionic Greens function is of the form

$$\langle 0|A|0\rangle = \frac{1}{Z} \int \mathcal{D}\psi \; \mathcal{D}\bar{\psi} \; A \; e^{-S_F}, \tag{119}$$

with an action S_F for the fermions. For a free Dirac field the action is

$$S_F = \int d^4x \; \bar{\psi}(x)(\gamma_\mu \partial^\mu + m)\psi(x). \tag{120}$$

In the context of the standard model, fermionic actions are always bilinear in the fermionic fields. With the help of the Grassmann integration rules above one can then show that the functional integrals are formally remarkably simple to calculate:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; e^{-\int d^4x \, \bar{\Psi}(x) Q \Psi(x)} = \det Q. \tag{121}$$

This is the famous fermion determinant. The main problem remains, of course, namely to evaluate the determinant of the typically huge matrix Q.

5.2 Naive Fermions

So far no difficulties for the implementation of fermions on the lattice seem to arise: all we have to do is to discretize the field configurations in the well-known way and to calculate the Greens functions with some of the methods of the last section. However, we fail. To see this, let us consider the propagator of a fermion with mass m as an example. The fermionic lattice action is then given by

$$S_F = \frac{1}{2} \sum_x \sum_\mu \bar{\psi}(x)(\gamma_\mu \Delta_\mu + m)\psi(x) + h.c. \tag{122}$$

and the resulting propagator is

$$\tilde{\Delta}(k) = \frac{-i \sum_\mu \gamma_\mu \sin k_\mu + m}{\sum_\mu \sin {k_\mu}^2 + m^2}. \tag{123}$$

The propagator has got a pole for small k_μ representing the physical particle, but there are additional poles near $k_\mu = \pm\pi$ due to the periodicity of the denominator. So S_F really describes 16 instead of 1 particle. This problem - euphemistically called fermion doubling - is a crucial obstacle for all lattice representations of quark fields and we shall next discuss some of the possible ways out of this dilemma.

5.3 Wilson vs. Staggered Fermions

Fermion doubling was already known to Wilson in the early days of lattice QCD. He proposed a modified action for the fermions in order to damp out the doubled fields in the continuum limit. Therefore he added another term, the Wilson term, to the naive action.

$$\begin{aligned} S_F \to S_F^{(W)} &= S_F - \frac{r}{2} \sum_x \bar{\psi}(x) \Box \psi(x) \\ &= S_F - \frac{r}{2} \sum_{x,\mu} \bar{\psi}(x) \{\psi(x+\hat{\mu}) + \psi(x-\hat{\mu}) - 2\psi(x)\}, \end{aligned} \tag{124}$$

where $0 < r \leq 1$. Calculating the propagator with this modified action, one finds that the unwanted doubled fermions acquire masses $\propto 1/a$, so that they become infinitely massive in the continuum limit and dissapear from the physical spectrum.

Wilson fermions have a serious disadvantage: even at vanishing fermion masses, chiral symmetry is broken explicitely by the Wilson term, and one

has problems with calculations for which chiral symmetry is of central importance.

There are alternatives to Wilson's approach. One of them, due to Kogut and Susskind, are so-called staggered fermions. The idea is to distribute the components ψ_α of the Dirac field on different lattice points. It results in a reduction from 16 to 4 fermions. Moreover, for massless fermions a remnant of chiral symmetry in form of a chiral U(1)$\otimes$U(1)-symmetry remains.

5.4 QCD on the Lattice

Neglecting the problem of chiral symmetry for the moment, we are now able to write down QCD for a discretized space-time. Take Wilson's or staggered fermions for the quark fields

$$\psi^f_{c\alpha}(x), \tag{125}$$

where $f = 1, \ldots, N_f$ and $c = 1, 2, 3$ are flavour and colour indices, and define the action as a sum of the pure gauge action for SU(3),

$$S_G = \beta \sum_p \left(1 - \frac{1}{3}\mathrm{ReTr}U(p)\right), \tag{126}$$

and the fermionic part S_F:

$$S = S_F + S_G. \tag{127}$$

Since computers cannot handle anticommuting numbers, in Monte Carlo calculations one performs the integration over the fermionic fields explicitly as in Eq. (121) and works with an effective action for the bosonic fields,

$$e^{-S_{eff}(U)} \equiv e^{-S_G(U)} \cdot \det Q(U), \tag{128}$$

which involves the fermion determinant. Because the calculation of $\det Q$ turns out to be very tedious, one often uses the quenched approximation that treats Q as a constant. In recent years different unquenched investigations of QCD have been made and have given estimates for quenching errors.

6 Physics

After having introduced the concepts and some of the technicalities, it is now time to turn to physics and to discuss physically relevant results. For this purpose it is necessary to talk about the way the continuum limit should be taken.

6.1 Continuum Limit

As we are only able to perform calculations at finite lattice spacing, it is an important issue to get the extrapolation process to the continuum limit under control. Since the lattice spacing is the regulator of our theory, it should be useful to apply renormalization group techniques to this problem. Knowing the functional dependence of the bare coupling g_0 on the regulator, in other words solving the renormalization group equation, we should know how to vary the bare coupling of our theory in order to reach a continuum limit. Let us discuss this idea in more detail.

In the continuum limit the lattice spacing a is supposed to go to zero, while physical masses m should approach a finite limit. The lattice spacing, however, is not a dimensionless quantity, therefore we have to fix some mass scale m, e.g. some particle mass, and consider the limit $am \to 0$. The inverse of that,

$$\frac{1}{am} \equiv \xi, \tag{129}$$

can be regarded as a correlation length. In the continuum limit ξ has to go to infinity, which is called a *critical point* of the theory. In Fig. 16 this is illustrated on a two-dimensional lattice with different correlation lengths.

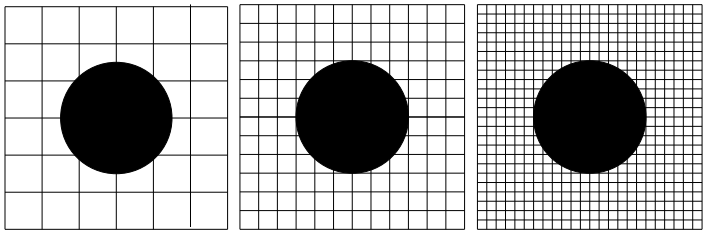

Figure 16: 2-dimensional lattices with increasing correlation lengths ξ

In pure gauge theory, there is a single, dimensionless bare coupling g_0 and am is clearly a function of g_0. In order to approach the continuum limit, we have to vary g_0 such that $am \to 0$. How this is done, is controlled by a renormalization group equation:

$$-a\frac{\partial g_0}{\partial a} = \beta_{LAT}(g_0) = -\beta_0 g_0^3 - \beta_1 g_0^5 + \ldots, \tag{130}$$

where the first term of the expansion is

$$\beta_0 = \frac{11}{3} N \frac{1}{16\pi^2}. \tag{131}$$

In the perturbative regime of g_0 this equation implies that for decreasing am the bare coupling g_0 is also decreasing, getting even closer to zero. Hence the continuum limit is associated with the limit

$$g_0 \to 0 \qquad \text{(continuum limit)}. \tag{132}$$

The solution of the renormalization group equation up to second order in g_0 is

$$a = \Lambda_{LAT}^{-1} \ \exp\left(-\frac{1}{2\beta_0 g_0^2}\right) \ (\beta_0 g_0^2)^{-\frac{\beta_1}{2\beta_0^2}} \ \{1 + \mathcal{O}(g_0^2)\}, \tag{133}$$

where the lattice Λ-parameter Λ_{LAT} appears. Solving for g_0 yields

$$g_0^2 = \frac{-1}{\beta_0 \log a^2 \Lambda_{LAT}^2} + \ldots, \tag{134}$$

which again reveals the vanishing of g_0 in the continuum limit:

$$g_0^2 \to 0 \quad \text{for } a \to 0. \tag{135}$$

We can also observe that

$$am = C \ \exp\left(-\frac{1}{2\beta_0 g_0^2}\right) \cdot (\ldots), \tag{136}$$

which shows the non-perturbative origin of the mass m.

These considerations, based on the perturbative β-function, motivate the following hypothesis: the continuum limit of a gauge theory on a lattice is to be taken at $g_0 \to 0$. Moreover, we expect that it involves massive interacting glueballs and static quark confinement.

The scenario for approaching the continuum limit then is as follows. Calculating masses in lattice units, i.e. numbers am, and decreasing g_0, we should reach a region where dimensionless quantities am follow a behaviour as given by Eq. (136). Plotting am logarithmically versus $1/g_0^2$ as in Fig. 17, an approximate linear behaviour is expected, which is called *asymptotic scaling*.

For mass ratios it can be shown that the exponential dependence on $1/g_0^2$ cancels out and that near the continuum limit

$$\frac{m_1}{m_2} = \text{const.} \times (1 + \mathcal{O}(a^2)). \tag{137}$$

Such a behaviour, $m_1/m_2 \approx$ const., is called *scaling*.

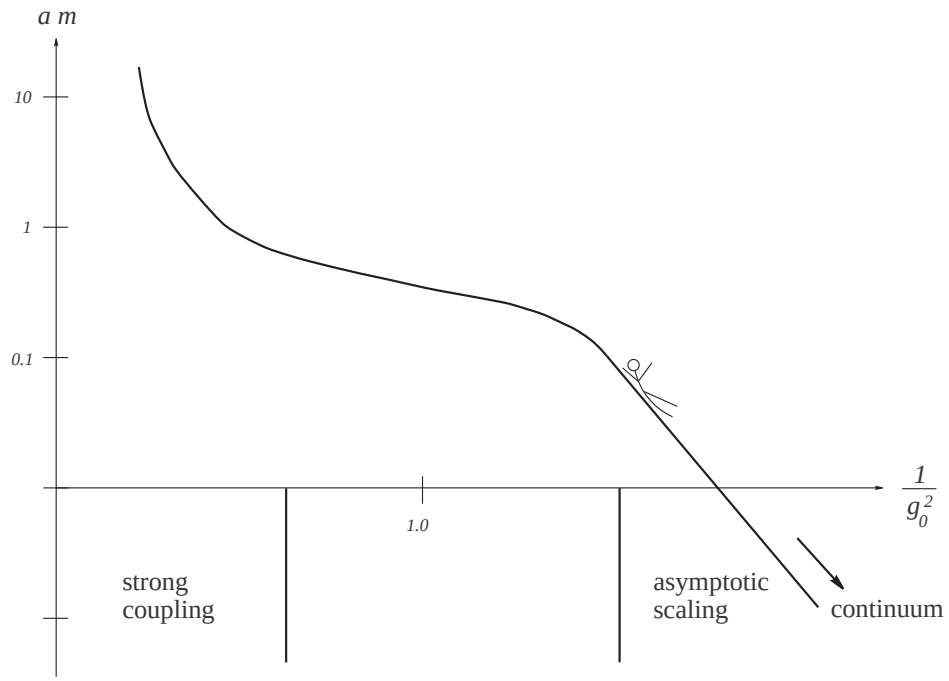

Figure 17: Scaling scenario

6.2 Error Sources

The results obtained by means of the Monte Carlo method differ from the desired physical results by different sorts of errors. The most important error sources are

- *statistical errors*: due to the finite number of configurations in the Monte Carlo calculation, $\sim 1/n^{1/2}$,
- *lattice effects*: due to finite lattice spacing a, often $\sim a$ or a^2,
- *volume effects*: due to finite lattice volume, often $\sim 1/L$, $1/L^2$, or e^{-mL},
- *large quark masses*: m_q mostly too big in Monte Carlo calculations,
- *quenched approximation*: $\det Q = 1$, neglecting the fermion dynamics.

6.3 QCD Calculations

In the lectures at Zuoz the methods, physical quantities and error sources introduced have been illustrated by various numerical results for lattice QCD, obtained by different groups in recent years. This includes results on the

- glueball spectrum,
- static quark potential,
- flux tubes between static quarks,
- hadron spectrum and decay constants,
- QCD running coupling α_S.

Since the corresponding figures can be found in the literature, we decided not to reproduce them here and instead refer to the literature [5, 1].

Other topics of physical interest, which have been investigated in lattice QCD, include hadronic and weak matrix elements, finite temperature QCD, quark-gluon plasma etc..

6.4 Higgs Models

As already mentioned in the beginning, non-perturbative studies of gauge theories are of interest in the electroweak sector as well. With Higgs models (pure gauge theory with scalar Higgs field) the electroweak phase transition has been studied. In Yukawa models (scalar fields and fermionic fields) the dynamical mass generation due to the coupling of Higgs fields to the fermions has been investigated in order to find bounds for the Higgs mass. Studies with Higgs-Yukawa models, including all three types of fields, have also been performed.

To conclude, a variety of lattice studies to extract parameters of the standard model and to clarify implications of the physics beyond have been made and are on their way.

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