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Introduction to Chiral Perturbation Theory*

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Abstract

An elementary introduction to chiral perturbation theory is given. The emphasis throughout is on basic ideas and the general structure, but not on formalism or recent developments. The mechanism of spontaneous chiral symmetry breaking is discussed in some detail. A scenario where the light quark condensate is assumed to be small is reviewed and possible phenomenological consequences are pointed out.

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1 Introduction

Chiral perturbation theory (ChPT) is the effective field theory of QCD at low energies. The perturbative series is organized in terms of small momenta and the light quark masses. The relevant expansion parameter is p/Λ_χ, where p denotes a generic low energy scale and $\Lambda_\chi \approx 1$ GeV is the scale of spontaneous chiral symmetry breaking. All this is made possible by the usual trick of effective field theories: only the degrees of freedom relevant at low energies are kept. In ChPT these degrees of freedom are lightest particles of the QCD spectrum, *i.e.* the pions. The fact that the pions are the (pseudo) Goldstone bosons of the spontaneously broken chiral symmetry yields severe constraints on the possible interactions at low energies. ChPT thus makes essential use of the global symmetries of QCD, and the predictions obtained are model independent.

Chiral symmetry has a long history. In the pre-QCD era it was realized that the strong interaction exhibit an approximate chiral symmetry, realized in the Nambu-Goldstone mode. This lead to the formulation of current algebra and PCAC (partially conserved axial-vector hypothesis) and culminated in the discovery of the famous soft-pion theorems. We shall give a very brief introduction to these topics in sections 2 and 3. For more details we refer to the reviews [1]. Chiral lagrangians were introduced first in [2] but used only at the tree level. Calculations of so-called leading chiral logs followed in refs. [3] but still no systematic framework was available. The breakthrough came when Weinberg showed that the loop expansion is equivalent to a low energy expansion. [4] The modern formulation of ChPT was developed in two seminal papers by Gasser and Leutwyler. [5, 6] After this many applications have been worked out in hundreds of papers. These range from strong interactions like $\pi\pi$-scattering or form factor calculations to nonleptonic weak interactions, from applications to finite temperature and finite volume to the baryonic sector of ChPT etc. I refer the reader to the existing excellent reviews, *e.g.* [7, 8, 9, 10]. Here I will be able only to sketch the basic ingredients and the structure of the theory.

Since this school is devoted to hidden or spontaneously broken symmetries, I have discussed in some detail possible mechanisms of spontaneous chiral symmetry breakdown (SχSB). In particular, in part II I shall describe a scenario where the quark condensate is *not* the leading order parameter. Although there are no contradictions to the picture where SχSB is triggered by a large quark condensate, there is presently also no phenomenological evidence against a picture with a much smaller condensate as put forward by Stern and collaborators. [11] The point is that this question can be decided phenomenologically by measuring the $I = 0$ s-wave scattering length to high precision [12], see section 6. The scenario has also interesting theoretical implications which will briefly be touched upon in section 7.

These lectures are hold on an elementary level and whenever possible the emphasis is put on the basic ideas and general structure of the theory. Also, due to lack of space, a rather limited selection of topics had to be made. I hope that nevertheless I have been able to convey some of the many facets *hidden* in low energy QCD.

Part I: Chiral perturbation theory

2 Chiral symmetry: basics

We start by considering the free theory of N_F fermions with lagrangian

$$\begin{aligned}\mathcal{L} &= \bar{\Psi}(i\,\partial\!\!\!/ - M)\Psi \\ &= \bar{\Psi}_L i\,\partial\!\!\!/\Psi_L + \bar{\Psi}_R i\,\partial\!\!\!/\Psi_R - \bar{\Psi}_L M\Psi_R - \bar{\Psi}_R M\Psi_L\ .\end{aligned} \tag{1}$$

Here Ψ denotes a column vector of N_F Dirac fermion fields and M is the diagonal quark mass matrix with positive entries $m_1 \ldots m_F$. The subscripts L, R denote the projection onto left and right handed components, respectively:

$$\Psi_{L,R} = P_{+,-}\Psi, \qquad P_{+,-} = \frac{1 \pm \gamma_5}{2}\ . \tag{2}$$

If we set the quark mass matrix M to zero, left and right handed components can be rotated independently without changing the lagrangian. The set of transformations

$$\Psi_R(x) \to e^{i\alpha_a\lambda^a}\Psi_R(x), \quad \Psi_L(x) \to e^{i\beta_a\lambda^a}\Psi_L(x) \tag{3}$$

with α_a, β_a constant, λ^a generators of $U(N_F)$ leaves the lagrangian (2) invariant. The theory thus exhibits the global chiral symmetry

$$U(N_F)_L \otimes U(N_F)_R \simeq U(1)_L \otimes U(1)_R \otimes SU(N_F)_L \otimes SU(N_F)_R\ . \tag{4}$$

The axial component $U(1)_A$ is actually anomalous in QCD.[13] This means that although the symmetry is present at the classical level, there is no consistent way of implementing it in the quantum theory. We will not discuss the anomalous sector in this lecture. The remaining, vectorial factor $U(1)_V$ is always conserved. The corresponding quantum number is the fermion number. We therefore consider here only the $SU(N_F)_L \otimes SU(N_F)_R$ part of the chiral symmetry. Associated to this chiral symmetry are the Noether currents

$$\begin{aligned}V_\mu^a(x) &= \bar{\Psi}(x)\gamma_\mu\lambda^a\Psi(x), \\ A_\mu^a(x) &= \bar{\Psi}(x)\gamma_\mu\gamma_5\lambda^a\Psi(x), \quad a = 1, \ldots, N_F^2 - 1\end{aligned} \tag{5}$$

as well as charges

$$Q_V(t) = \int d^3x V_0(x), \quad Q_A(t) = \int d^3x A_0(x)\ . \tag{6}$$

In the massless case these currents are conserved and the corresponding charges are constants of the motion.

Switching on the quark masses, the vector and axial vector currents are no longer conserved but obey

$$\begin{aligned}\partial^\mu V_\mu^a(x) &= i\bar{\Psi}(x)\,[M, \lambda^a]\,\Psi(x) \\ \partial^\mu A_\mu^a(x) &= i\bar{\Psi}(x)\,\{M, \lambda^a\}\,\Psi(x)\ .\end{aligned} \tag{7}$$

A few comments are in order:

- if all masses are the same, *i.e.* M is a multiple of the unit matrix, all vector currents are conserved. For $N_F = 2, 3$ this limit is usually referred to as the $SU(2)$ or $SU(3)$ limit.
- the axial-vector currents are broken for $M \neq 0$.
- if finally $M = 0$, then both, vector as well as axial-vector currents are conserved. This limit is referred to as the chiral limit.

The simple lagrangian of Eq. (2) was used to exemplify the immediate consequences of a chiral symmetry. However, in the 60'ties it became clear that the strong interactions respect to a good approximation this global chiral symmetry. It was then conjectured that the the associated currents obey the so-called current algebra [1]

$$\begin{aligned}
\left[Q_V^a(x_0), V_\mu^b(y)\right]\delta(x_0 - y_0) &= if_{abc}V_\mu^c(y) \\
\left[Q_V^a(x_0), A_\mu^b(y)\right]\delta(x_0 - y_0) &= if_{abc}A_\mu^c(y) \\
\left[Q_A^a(x_0), V_\mu^b(y)\right]\delta(x_0 - y_0) &= if_{abc}A_\mu^c(y) \\
\left[Q_A^a(x_0), A_\mu^b(y)\right]\delta(x_0 - y_0) &= if_{abc}V_\mu^c(y),
\end{aligned} \tag{8}$$

where f_{abc} denote the structure constants of $SU(3)$.

We close this section with a simple example showing how the current algebra can be used to obtain non-trivial relations between seemingly unrelated matrix-elements. Consider the vector form factor of the pion,

$$< \pi^+(p')|V_\mu^3(0)|\pi^+(p) >= F_1(q^2)(p+p')_\mu, \quad q = p - p' \ . \tag{9}$$

A second form factor in proportion to q_μ is absent due to current conservation. Now we employ the current algebra (8) in the form

$$V_\mu^3(0) = \frac{1}{2}\left[Q_V^+, V_\mu^-\right], \qquad Q_V^+ = Q_V^1 + iQ_V^2 \ . \tag{10}$$

Moreover, we notice that the spectrum must be given by irreducible representations of $SU(3)_V$, hence

$$< \pi^+|Q_V^+ = \sqrt{2} < \pi^0|, \qquad Q_V^+|\pi^+ >= 0. \tag{11}$$

We thus obtain

$$< \pi^+(p')|V_\mu^3(0)|\pi^+(p) >= \frac{1}{\sqrt{2}} < \pi^0(p')|V_\mu^-(0)|\pi^+(p) >, \tag{12}$$

where the matrix element on the RHS of Eq. (12) is related to pion β-decay. Notice that in this simple example only the vectorial part of the current algebra (8) was used.

3 Chiral symmetry in the Nambu-Goldstone mode

We have argued that the strong interactions exhibit a global chiral symmetry $SU(N_F)_L \otimes SU(N_F)_R$. However, this symmetry cannot be realized in the usual manner, for then the spectrum should also exhibit this symmetry. More precisely, consider a Hamiltonian $\mathcal{H}_0$ symmetric under the chiral group. This Hamiltonian in particular commutes with the generators of the axial transformations

$$[\mathcal{H}_0, Q_A^a] = 0 \ . \tag{13}$$

If $|\psi>$ is an eigenstate of $\mathcal{H}_0$ with definite parity, *i.e.*

$$\mathcal{H}_0|\psi>= E|\psi>, \quad \mathcal{P}|\psi>= P|\psi>, \tag{14}$$

then there exists another state, $|\psi_A>= Q_A^a|\psi>$, with the same energy and the opposite parity. The first statement is shown with the help of (13) whereas the second statement follows from the fact that the charges Q_A^a have negative parity. For each particle in the spectrum we therefore should find a partner with the same mass but opposite parity. This is obviously not true in our world: there are no parity even partners of the lightest particles, the pions, kaons, and the eta.

3.1 Goldstone's theorem

The paradox is resolved by noticing that there is another way to realize the symmetry, namely the Nambu-Goldstone mode. In order to understand the difference to the so far considered Wigner-Weyl mode, we need to introduce the concept of a spontaneously broken or hidden symmetry. Consider a theory with hamiltonian $\mathcal{H}$ invariant under the Lie group G. Thus

$$[\mathcal{H}, Q_i] = 0, \quad i = 1, \ldots, n_G, \tag{15}$$

where the Q_i are the generators of the group G. We then say the *symmetry is spontaneously broken* if the ground state is not invariant under G

$$Q_i|0>\neq 0, \quad i = 1, \ldots, n_G - n_H \ . \tag{16}$$

The states $Q_i|0>$ have the same energy as the ground state, *i.e.* $E = 0$. Moreover, Goldstone's theorem tells us that in a relativistically invariant theory this implies the existence of massless particles.[14] These are the Goldstone bosons. The n_H unbroken generators form a subgroup H, whereas the $n_G - n_H$ broken generators are elements of the coset space G/H. Each of these broken generators generates a linearly independent state of energy $E = 0$ when acting on the vacuum

$$Q_i|0>= |\pi_i> \ . \tag{17}$$

There are therefore $n_G - n_H$ Goldstone bosons in the spectrum of the theory.

In QCD the broken generators must be the axial charges. For $N_F = 3$ light flavors there are $N_F^2 - 1 = 8$ Goldstone bosons. These are the pions, the kaons, and the eta. The Goldstone theorem also implies that there is a constant $F_\pi \neq 0$ such that

$$< 0| \left[Q_A^a, \pi^b\right] |0>= \delta^{ab} F_\pi \tag{18}$$

and

$$< 0|A^a_\mu(0)|\pi^b(q) >= iF_\pi q_\mu \delta^{ab} \; . \tag{19}$$

An immediate and important consequence of the Goldstone theorem is that the creation of soft pions ($p_\mu = 0$) does not cost energy. The commutators involving the axial charges Q^a_A, *e.g.*

$$\left[Q^a_A(x_0), V^b_\mu(y)\right] \delta(x_0 - y_0) = if_{abc} A^c_\mu(y) \tag{20}$$

thus relate processes with different numbers of pions. These are the celebrated soft-pion theorems.[1]

3.2 Pion pole dominance

Related to this observation is the concept of pion pole dominance or partially conserved axial-vector current hypothesis. The idea is that in a theory with spontaneously broken symmetry, the singularities in the low energy region (one-particle poles, cuts) are entirely given by the Goldstone bosons. All other effects due to massive states or interactions can be neglected. At sufficiently low energy only the one-particle singularities, *i.e.* the pion-poles have to be taken into account.

A succesful application of this concept is provided by the Goldberger-Treiman relation.[15] Consider the matrix element of the axial-vector current between nucleon states

$$< N(p')|A^+_\mu(0)|N(p) >= \bar{u}(p') \left[\gamma_\mu g_A(q^2) + q_\mu g_3(q^2)\right] \gamma_5 \frac{\tau^+}{2} u(p), \tag{21}$$

where $q = p - p'$ is the momentum transfer. Current conservation and the Dirac equation imply

$$0 = 2m_N g_A(q^2) + q^2 g_3(q^2) \; . \tag{22}$$

Considering now the low energy limit we have

$$g_A(0) = -\frac{1}{2m_N} \lim_{q^2 \to 0} q^2 g_3(q^2) \; . \tag{23}$$

The nucleon axial coupling constant can be nonzero only if the form factor $g_3(q^2)$ has a pole at $q^2 = 0$. The pion yields such a pole contribution to the matrix element. Keeping only this term (pion pole dominance) we obtain

$$< N(p')|A^+_\mu(0)|N(p) >_{\text{pion pole}}= iq_\mu F_\pi \frac{1}{q^2} 2g_{\pi N} i\bar{u}(p')\gamma_5 \frac{\tau^+}{2} u(p), \tag{24}$$

where $g_{\pi N}$ is the pion-nucleon coupling constant. Combining (24) with (23) finally yields the Goldberger-Treiman relation

$$F_\pi g_{\pi N} = g_A(0) m_N \; . \tag{25}$$

This relations predicts F_π correctly to approximately 5 %. Mass corrections can be calculated with the aid of chiral perturbation theory.

4 QCD at low energies: Chiral perturbation theory

The discussion of chiral symmetry so far has shown the general structure of a theory in the chiral limit. Also, the concept of spontaneous breakdown of a global symmetry was introduced and shown to have important consequences. We now come to the topic we are mostly interested in: QCD at low energies. The QCD lagrangian has an exceedingly simple form

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{2g^2}\mathrm{tr}\, G^a_{\mu\nu}G^{a,\mu\nu} + \bar{q}i\gamma_\mu\left(\partial^\mu - iG^\mu\right)q - \bar{q}Mq \; . \tag{26}$$

For simplicity we have suppressed gauge fixing and source terms. Despite it's simple form, we do not know how to solve the problem, i.e. to give the set of Green functions of all possible external sources. There is no general analytical method available which would make possible to quantitatively predict *e.g.* the masses of the lightest particles or the leading order parameters like the light quark condensate. The reason is the following: at low energies, the QCD coupling constant α_s becomes strong and any perturbative treatment must break down. As long as we are not able to solve the nonperturbative problem, we must look for some other small expansion parameter. Chiral perturbation theory is such a scheme which exploits the symmetry properties of the theory in the limit of vanishing quark masses (chiral limit). Consider the lagrangian (26) for three light flavours,

$$N_F = 3, \qquad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} . \tag{27}$$

If the quark mass matrix is set to zero, the lagrangian (26) exhibits the global chiral symmetry $SU(3)_L \otimes SU(3)_R$. The quark masses are assumed to be small such that the symmetry limit is a good starting point. Eventually explicit symmetry breaking terms will be added – the corresponding small parameter will therefore be the quark mass. Moreover, we assume the symmetry to be spontaneously broken. As we have seen, this implies the existence of Goldstone bosons, which in turn interact weakly for soft momenta. The soft pion theorems mentioned in the previous section represent the symmetry limit of very low energy or momenta. The corresponding small expansion parameters are then the energy and momenta of the Goldstone bosons.

4.1 Construction of the chiral lagrangian

The program of chiral perturbation theory is then the following: Construct an effective low energy theory with global symmetry properties of QCD. More precisely, the constraints to be satisfied are

- The effective lagrangian must be invariant under global $SU(3)_L \otimes SU(3)_R$. It must also be invariant under Lorentz transformations as well as under the discrete symmetries P and C.

- Assume spontaneous chiral symmetry breaking such that the diagonal subgroup $SU(3)_V$ is unbroken. The broken generators Q^a_A imply the existence of eight Goldstone bosons.

- Retain only the massless particles in the effective theory. In the present case these are the pions, the kaons, and the eta.
- Organize the theory according to the number of derivatives. Terms with more derivatives will be suppressed by additional factors of the small momenta.
- Account for explicit symmetry breaking by expanding in small quark masses. This will allow to calculate the corrections to the symmetry limit.

This problem has an elegant mathematical solution. [16] We denote by G and H the full symmetry group and it's unbroken subgroup, respectively,

$$G = SU(3)_L \otimes SU(3)_R, \qquad H = SU(3)_V \; . \tag{28}$$

The elements we are interested in are those of the coset space G/H. According to [16] the action of G on G/H defines a non-linear realization $h(\phi)$:

$$\begin{aligned} & u(\phi) \xrightarrow{G} g_R u(\phi) h(\phi)^\dagger = h(\phi) u(\phi) g_L^\dagger \\ & u(\phi) \in G/H, \qquad g_{L,R} \in SU(3)_{L,R} \; . \end{aligned} \tag{29}$$

The transformation properties (29) imply that the square of $u(\phi)$ transforms linearly

$$U(\phi) \xrightarrow{G} g_R U(\phi) g_L^\dagger, \qquad U(\phi) = u^2(\phi) \; . \tag{30}$$

Thus we may construct invariants by multiplying the field U with it's hermitean conjugate and performing a trace in flavour space. However, since U is unitary, the simplest invariant of this form is just a constant. In order to construct non-trivial invariants, we take derivatives of the U-fields. Some examples of invariants are

$$\begin{aligned} & \mathrm{tr}\, \partial_\mu U^\dagger \partial^\mu U \\ & \mathrm{tr}\, \partial_\mu U^\dagger \partial_\nu U \partial^\mu U^\dagger \partial^\nu U \; \ldots \end{aligned} \tag{31}$$

The matrix U contains the Goldstone boson fields. A representation wich is often used is the so-called matrix representation

$$U = e^{i\phi/F_0}, \quad \phi = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \tag{32}$$

where F_0 is the pion decay constant in the chiral limit.

An important consequence of this construction is the fact that Goldstone bosons couple with derivatives. In momentum space the derivatives on any field translate into external momenta. The selfinteractions of the Goldstone bosons therefore scale with external momenta and, in the limit where these are taken soft, the interactions vanish. This is the reason why a perturbative treatment of low energy QCD is possible: once the collective coordinates (the pions) are chosen as degrees of freedom, the theory becomes weakly coupled at low energies!

The explicit symmetry breaking terms finally are those having the same transformation properties than the mass term in the QCD lagrangian (26), *i.e.*

$$\bar{q}Mq \rightarrow \mathrm{tr}\left(MU^\dagger + U^\dagger M\right) \tag{33}$$

Collecting the relevant terms we obtain the leading order chiral lagrangian

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4}\left\{\mathrm{tr}\,\partial_\mu U^\dagger \partial^\mu U + 2B_0 \mathrm{tr}\left(MU^\dagger + U^\dagger M\right)\right\} \ . \tag{34}$$

This lagrangian is also called a non-linear σ-model. It contains two parameters, the pion decay constant in the chiral limit, F_0, and the parameter B_0 which is related to the light quark condensate in the chiral limit

$$B_0 = -\frac{<\bar{q}q>}{F_0^2} \ . \tag{35}$$

We shall discuss the meaning of this second parameter in more detail in part II.

4.2 Consequences of $\mathcal{L}^{(2)}$

The effective lagrangian (34) can be used directly to evaluate matrix elements of processes involving any number of (pseudo) Goldstone bosons at tree level. If we wish to discuss the couplings to external sources the partial derivatives have to be replaced by covariant derivatives etc., see [6] for details. The lagrangian then embodies all low energy theorems (LET) which can be derived using current algebra and PCAC. In addition, all leading order mass-corrections to the abovementioned LET's can be worked out in this manner. The representation of all this information about low energy QCD is not only very compact, but also very efficient. The algebra necessary to work out the relevant Feynman graphs is much less involved than the manipulations one has to perform when using current algebra and PCAC.

In general, only three steps have to be taken for a given process:

1. expand the lagrangian in terms of the ϕ-fields $(U(\phi) = 1 + i\phi/F_0 - \phi^2/(2F_0^2) + \ldots)$
2. perform flavour traces and derive all relevant Feynman rules
3. work out the tree graphs contributing to a given process (contact and pole graphs)

Two simple applications of this sort follow. First we consider the masses of the pseudo Goldstone bosons. Expanding the lagrangian up to terms of $\mathcal{O}(\phi^2)$ one obtains for $m_u = m_d$, $\hat{m} = (m_u + m_d)/2$,

$$\begin{aligned} M_\pi^2 &= 2\hat{m}B_0 \\ M_K^2 &= (\hat{m} + m_s)\, B_0 \\ M_\eta^2 &= \frac{2}{3}(\hat{m} + 2m_s)\, B_0 \ . \end{aligned} \tag{36}$$

Combining the three equations (36) yields the Gell-Man-Okubo mass formula and the Weinberg ratio of quark masses [17], respectively

$$\begin{aligned} 4M_K^2 - M_\pi^2 - 3M_\eta^2 &= 0 \\ \frac{m_s}{\hat{m}} &= \frac{2M_K^2 - M_\pi^2}{M_\pi^2} \ . \end{aligned} \tag{37}$$

Both results depend on the assumption that the light quark condensate is large. The generalization to a possible small quark condensate will be discussed in part II.

Another famous result which can easily be derived from (34) is the Weinberg formula for low energy pion-pion scattering.[18] In this case the lagrangian has to be expanded up to terms of $\mathcal{O}(\phi^4)$, yielding

$$A(\pi^a\pi^a \to \pi^b\pi^b) = \frac{1}{F_0^2}\left(s - M_\pi^2\right) \ . \tag{38}$$

As promised, the amplitude vanishes in the limit of zero external momenta and quark masses.

4.3 Loop expansion and higher chiral orders

So far we have treated only tree graphs. These graphs can never account for unitarity for there is no way to generate imaginary parts by a tree amplitude. In the language I was using above we can also say that the only singularities tree graphs can account for are the single particle poles. Multiparticle singularities like cuts and branch points will show up only if one includes loop graphs. However, the non-linear σ-model is nonrenormalizable and loop graphs in general will introduce divergences. Moreover, at first sight it is not clear how loop corrections can be included in the effective field theory.

The point is that chiral perturbation theory allows for a *systematic loop expansion*. In an important paper [4] Weinberg showed that the loop expansion is equivalent to a low energy expansion. In order to maintain a consistent effective field theory framework one has to include all possible higher derivative terms allowed by the symmetry constraint. Thus the effective lagrangian is written

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots \tag{39}$$

where the superscripts denote the number of derivatives. As remarked earlier, this number can also be considered as a low energy dimension, *i.e.* a term of $\mathcal{L}^{(n)}$ counts as $\mathcal{O}(p^n)$, where p denotes a generic low energy momentum. Weinberg showed that the low energy dimension D of any Feynman graph is given by

$$D = 2 + \sum_d N_d(d-2) + 2N_L \tag{40}$$

where N_d is the number of vertices with d derivatives and N_L is the number of loops. We notice that any further loop added enlarges the low energy dimension of a given graph by 2 units. The relevant graphs one has to consider up to $\mathcal{O}(p^6)$ are thus

$\mathcal{O}(p^2)$ tree graphs with any number of vertices from $\mathcal{L}^{(2)}$

$\mathcal{O}(p^4)$ one-loop graphs with any number of vertices from $\mathcal{L}^{(2)}$ as well as tree graphs with at most one vertex from $\mathcal{L}^{(4)}$

$\mathcal{O}(p^6)$ two-loop graphs with any number of vertices from $\mathcal{L}^{(2)}$, one-loop graphs with at most one vertex from $\mathcal{L}^{(4)}$ as well as tree graphs with either at most one vertex from $\mathcal{L}^{(6)}$ or with two vertices from $\mathcal{L}^{(4)}$

In two seminal papers [5, 6] Gasser and Leutwyler have taken up these ideas and put on a firm ground. In their formulation, they consider directly the generating functional of Green functions in the low energy regime. This is not only crucial for the foundation of chiral perturbation theory [19] but also has many advantages in practical applications. Also, renormalization to one-loop was treated in full generality. Here we must content ourselves with a few remarks but refer the interested reader to the original literature cited above.

The inclusion of the lagrangians with higher order derivative terms is essential. For one the loop graphs in general are divergent. Since loops augment the low energy dimension, these divergencies cannot possibly be absorbed into renormalized coupling constants of the leading order lagrangian. The coupling constants of the higher order lagrangians on the other hand can do the job. Since they are the most general structures allowed by chiral symmetry it is actually guaranteed that the renormalization is always feasible. Second, the finite parts of these higher order coupling constants have to be kept too. There is no reason why a term compatible with the symmetry should be discarded, even if its coupling constant does not get renormalized. In fact these terms have a well defined physical meaning: they replace all the heavy degrees of freedom which have not been kept explicitly in the theory by local terms. This situation is typical for a non-renormalizable effective field theory. [20]

As long as we have no control over the underlying theory we have to treat the coupling constants of the higher order lagrangians as free parameters. Eventually they will be fixed by comparison of the calculations with experimental data. When going to higher order in the loop expansion, more and more free coupling constants will be entering the calculations and the predictive power of the scheme will in principle be lost. However, the low energy expansion converges quickly enough such that one needs to include only the next-to-leading order contributions to a given process. The data then overdetermine the coupling constants of the chiral lagrangian and consequently non-trivial relations between differen processes follow.

In order to illustrate these comments I consider the electro-magnetic form factor of the pion to one-loop. Define

$$< \pi^+(p')|V_\mu|\pi^+(p) >= (p_\mu + p'_\mu) f_{\pi^+}(t), \quad t = (p'-p)^2 \; . \tag{41}$$

At leading order p^2, there is only one tree graph contributing. At next to leading order, two distinct contributions appear: one-loop graphs with intermediate two-pion and two-kaon states and a contact term. The contact term arises from the lagrangian $\mathcal{L}^{(4)}$. The entire lagrangian consists of 12 terms [6], but for the present application only the L_9-term contributes

$$\mathcal{L}^{(4)} = \ldots + -iL_9 \mathrm{tr}\left(F^R_{\mu\nu}\partial^\mu U \partial^\nu U^\dagger + F^L_{\mu\nu}\partial^\mu U^\dagger \partial^\nu U\right) + \ldots \tag{42}$$

The pion form factor to one-loop is then given as [21]

$$f_{\pi^+}(t) = 1 + \frac{t}{F_0^2}\left\{2\phi_\pi(t) + \phi_K(t) + 2L_9^r\right\} \ . \tag{43}$$

The functions $\phi_P(t)$ are known one-loop integrals which exact form is of no importance for the following discussion. They contain cuts starting at $4M_\pi^2$ and $4M_K^2$ generated by the two-pion and two-kaon intermediate states, respectively. The contact term in proportion to L_9^r is a priori not known. If we want to predict the electro-magnetic charge radius, *e.g.*, we need the value of this renormalized low energy constant. The point is that L_9^r can be fixed from the slope of a K_{l3} form factor. Inserting this number in (43) one obtains for the charge radius

$$< r^2 >_{\pi^+} = \left\{ \begin{array}{ll} 0.42\,\mathrm{fm}^2 & \mathrm{one-loop} \\ 0.44\,\mathrm{fm}^2 & \mathrm{exp.} \end{array} \right. \tag{44}$$

which compares well with the experimentally known number.

For the case at hand the loop graphs are actually strongly suppressed. The main effect comes from the coupling constant L_9^r. This fact can be understood by studying the physics which governs the coupling constant L_9^r. It turns out that there is a strong resonance in the channel we are considering, the ρ-meson. We can think of L_9^r as representing the effect of the ρ-meson at low energies. The corresponding quantitative estimate (so-called resonance saturation) yields

$$L_9^r = \frac{F_V^2}{2M_\rho^2} \tag{45}$$

which is in reasonable agreement with the experimental determination leading to (44). We conclude that the leading singularity in this channel is given by the single-particle pole of the ρ – the two-particle cuts play only a minor role. Since the chiral lagrangian contains the effects of heavier states encoded in the low energy constants like L_9^r, such important effects are included. The other constants of the $\mathcal{O}(p^4)$ lagrangian can also be understood with the principle of resonance saturation. [22]

Part II: Mechanism of chiral symmetry breaking

The formulation of chiral perturbation theory in part I was based on the assumption that chiral symmetry is spontaneously broken in QCD. The only argument in favour of this scenario we used was purely phenomenological: the spectrum of the light hadrons and the interactions thereof are such that this particular pattern of chiral symmetry breaking must occur. If not, QCD were not able to explain the world as we know it.

What do we know about chiral symmetry breaking from the theoretical point of view? In this second part we shall investigate two possible scenarios which seem to be viable. A useful concept that will be introduced is the notion of an order parameter. A central question to be asked is whether the condensate of light quarks is the leading order parameter of spontaneous breakdown of chiral symmetry. Although this is widely believed to be true, it will be argued that neither phenomenology nor theoretical considerations can presently rule out a scenario with vanishing or small quark condensate. There is a good chance to decide the issue by measuring the $\pi\pi$-scattering length with $J = I = 0$ to high precision, however. On the theoretical side, studies of the spectrum of the Euclidean Dirac operator in the background of gauge fields on the the lattice should also yield clean signals for either of the two scenarios.

Before we discuss these issues in more detail, we review what is actually known about spontaneous chiral symmetry breakdown in QCD. Two important observations are

1. In vector-like gauge theories like QCD (with vacuum angle $\theta_{\rm QCD} = 0$), the generators of vector symmetries Q^a_V remain unbroken. [23]

2. Assuming colour confinement (no coloured states in the spectrum), the anomaly matching conditions imply the spontaneous breakdown of the axial part of chiral symmetry for $N_F \geq 3$. [24, 25] In the proof of this statement it is assumed that the chiral Ward identities are not modified by confinement.

These two statements, taken together with the Goldstone theorem, predict the existence of the pions, kaons, and the eta in QCD under rather general assumptions! However, the proof gives no information on the mechanism of SχSB.

5 Order parameters of QCD

The concept of an order parameter is well known from solid state or statistical physics. Consider a spin system with either Ferromagnetic or Anti-Ferromagnetic order. Above a critical temperature T_c, the spins are randomly oriented and the magnetization vanishes. If the system is cooled down below T_c long range order sets in. In the Ferromagnetic case, the spins are aligned and the magnetization is finite. In the Anti-Ferromagnetic case, the spins also aligne along a certain direction, but adjacent spins have opposite direction. Although there is a long range order, the magnetization is zero. The relevant order parameter in this case is the so-called "staggered magnetization", where in the sum over spins every second entry is counted with a negative sign. We shall keep in mind these two situations as an analogue to the scenarios we shall discuss for QCD.

Next we need a more precise definition of an order parameter. Consider a system which is symmetric under the action of a continuous group G. An order parameter is any operator O that transforms under G as the sum of irreducible representations which however does not include the unit representation. If such an order parameter has non-vanishing vacuum expectation value

$$< 0|O|0 > \neq 0, \tag{46}$$

the symmetry is spontaneously broken. The converse is not true, however. If the symmetry is spontaneously broken, not any operator with nontrivial transformation property under G has non-vanishing vacuum expectaton value.

In order to illustrate this last point consider again a system with Ferromagnetic order. Assume the hamiltonian to be rotationally invariant. Since the magnetization transforms as a vector under rotations, the components can be written as

$$M_i = -\frac{i}{2}\epsilon_{ijk}[L_j, M_k] \ . \tag{47}$$

If in addition the ground state is rotationally invariant, we have

$$L_i|0>=0 \quad \longrightarrow \quad <0|M_i|0>=0. \tag{48}$$

Thus $<0|M_i|0>\neq 0$ necessarily implies $L_i|0>\neq 0$, *i.e.* spontaneous breakdown of rotational symmetry. In general there will be an infinite number of order parameters. Another example is provided by

$$O_{ij} = M_i M_j - \frac{1}{3}\vec{M}^2\delta_{ij} \tag{49}$$

which is purely $J=2$. A nonvanishing vacuum expectation value of O_{ij} again implies SSB. However, it is possible to simultaneously have $<0|M_i|0>=0$.

The order parameters of QCD are the composite operators with nontrivial transformation properties under the chiral group. Obviously we consider only operators that are allowed by the discrete transformations P, C and T. The quark fields transform under $SU(3)_L \otimes SU(3)_R$ as

$$u_L,\ d_L,\ s_L: \quad (3,1), \qquad u_R,\ d_R,\ s_R: \quad (1,3) \ . \tag{50}$$

Simple examples of composite operators and their transformation properties under the chiral group are then

$$\begin{aligned} \bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L &\simeq (3^*,3)\oplus(3,3^*) \\ V^a_\mu, A^a_\mu = J^a_{\mu,L} \pm J^a_{\mu,R} &\simeq (8,1)\oplus(1,8) \ . \end{aligned} \tag{51}$$

Finally, we enumerate the most important order parameters in QCD and discuss some of it's features. These are

1. The quark condensate $<0|\bar{q}q|0>$. It is the simplest order parameter and constitutes the leading order term in the expansion of the meson mass squared. In the standard scenario of ChPT the quark condensate is assumed to be large.

2. The two-point function

$$\Pi^{ab}_{\mu\nu} = \int d^4x e^{iqx} <0|T\left(V^a_\mu(x)V^b_\nu(0) - A^a_\mu(x)A^b_\nu(0)\right)|0>$$

   is related to the pion decay constant by $\Pi^{\mu,ab}_\mu(0) = 3iF_0^2\delta^{ab}$. It is the only order parameter which yields a *necessary and sufficient condition for* SχSB.

3. The two-point function

$$\Pi^{ab}_{SP} = \frac{i\delta^{ab}}{F_0^2}\int d^4x e^{iqx} <0|T\left(S^a(x)S^b(0) - P^a(x)P^b(0)\right)|0>$$

   of scalar and pseudo scalar densities is related to the order parameter A_0. This order parameter governs the term quadratic in quark mass in the expansion of the meson mass squared (see below).

6 Generalized versus standard ChPT

In the formulation of chiral perturbation theory as given in [5, 6] the light quark condensate is assumed to be large. The question whether the quark condensate is really the leading order parameter of SχSB cannot be answered within this theory – only consistency checks are possible. Stern and collaborators have set up a framework which allows to test this assumption: Generalized ChPT.[11] The basic idea can best be understood by considering the quark mass expansion of the pion mass squared [1]

$$M_\pi^2 = 2B_0\hat{m} + 4A_0\hat{m}^2 + \text{chiral logs} + \mathcal{O}(\hat{m}^3) \ . \tag{52}$$

If both, B_0 and A_0 are of the same order of magnitude, the first term on the right hand side dominates due to the additional quark mass suppression of the A_0-term. This is the *standard scenario* where

$$\begin{aligned}&\text{Standard ChPT}\\ &< \bar{q}q > \approx -(230\,\text{MeV})^3 \longrightarrow B_0 \approx 1.2\,\text{GeV} \longrightarrow m_q \simeq \mathcal{O}(p^2) \ .\end{aligned} \tag{53}$$

According to (52) the quark mass is then counted as a quantity of order p^2.

In the *Generalized scenario*, on the other hand, the quark condensate is allowed to be small or even vanishing. Typical values for the parameter B_0 are

$$\begin{aligned}&\text{Generalized ChPT}\\ &< \bar{q}q > \approx -(0 \ldots 100\,\text{MeV})^3 \longrightarrow B_0 \approx 0 \ldots 200\,\text{MeV} \longrightarrow m_q \simeq \mathcal{O}(p) \ .\end{aligned} \tag{54}$$

The A_0-term in (52) can then be as large as the B_0-term. In the case where the quark condensate vanishes, A_0 is actually the leading order parameter. This point of view leads to a different low-energy counting of the quark masses. Since B_0 is taken as a small quantity of order p, the quark mass must have low energy dimension p too. Consequently, in Generalized ChPT the effective lagrangian is written

$$\mathcal{L} = \sum_{k\geq 2, l\geq 0} \mathcal{L}_{k,l}, \qquad \mathcal{L}_{k,l} \simeq \mathcal{O}(p^{k+l}), \tag{55}$$

where k is the number of derivatives and l the number of quark mass insertions. The main differences to the standard ChPT are

- the standard ChPT series is reordered
- mass terms appear at lower order, *i.e.* more terms contribute at a given order of the low energy expansion
- successive orders in the perturbation series are suppressed by only one power of low energy momenta

[1] This relation is remarkable since it shows that in order to give the pion mass, both, eplicit ($\hat{m} \neq 0$) as well as spontaneous (nonzero order parameters) breakdown of chiral symmetry is needed.

Extending these ideas to chiral SU(3), it can also be shown that the quark mass ratio $m_s/\hat{m}$ must be in the range

$$\frac{2M_K}{M_\pi} - 1 \leq \frac{m_s}{\hat{m}} \leq \frac{2M_K^2}{M_\pi^2} - 1 \ . \tag{56}$$

The lower bound corresponds to a vanishing quark condensate. Numerically, this corresponds to $m_s/\hat{m} = 6.3$ which has to be compared to the upper bound $m_s/\hat{m} = 25.9$.

6.1 Phenomenological tests: $\pi\pi$–scattering

Since the difference between standard and generalized ChPT shows up only in the explicit symmetry breaking terms in proportion to quark masses, the effects are in general small. There is one prominent example where the effect is substantial: $\pi\pi$-scattering at low energies. Neglecting isospin breaking the scattering amplitude is given in terms of one amplitude $A(s,t,u)$. To leading order p^2 generalized ChPT yields

$$A(s,t,u) = \frac{\beta}{M_\pi^2}(s - \frac{4}{3}M_\pi^2) + \alpha\frac{M_\pi^2}{3F_\pi^2} \tag{57}$$

where

$$\alpha = 4 - 3\frac{2\hat{m}B_0}{M_\pi^2} + \mathcal{O}(m_q), \quad \beta = 1 + \mathcal{O}(m_q) \ . \tag{58}$$

Whereas the parameter β stays close to 1, α varies strongly with the quark condensate. For large B_0 we have $\alpha = 1$ and thus recover the Weinberg formula (38). For vanishing condensate, *i.e.* the extreme generalized case, we find $\alpha = 4$. This means that the amplitude at the symmetric point is 4 times larger if the condensate is zero. Translated to the $I = 0$ s-wave scattering length we find

$$\begin{aligned} a_0^{0,\,\mathrm{Standard}} &= \frac{M_\pi^2}{4\pi F_\pi^2} = 0.16 \\ a_0^{0,\,\mathrm{Generalized}} &= \frac{39}{24} a_0^{0,\,\mathrm{Standard}} = 0.26 \ . \end{aligned} \tag{59}$$

Both, one- and two-loop corrections have been worked out for the two scenarios.[12] In the standard case, the scatteting length moves to $a_0^0 = 0.20$ at one-loop and $a_0^0 = 0.217$ at two loop, with a small error bar of ± 0.01. The corrections for small quark condesate are somewhat smaller, but the general picture found already at tree level remains: a scattering length of $a_0^0 = 0.26$ would clearly signal a substantially smaller condensate than what is assumed in the standard scenario of SχSB (see ref. [12] for details). There is hope that this important issue can be decided in the near future at the high precision experiments E865 (Brookhaven), KLOE (DaΦne) or DIRAC (CERN).

7 Theoretical Ideas of SχSB

Finally I would like to briefly comment on some theoretical determinations of the light quark condensate. These are

- Determinations based on QCD sum rules tend to obtain large condensates [26] and thus are often cited as evidence against the small condensate scenario. However, the Gell-Man-Oakes-Renner relation is often implicitly assumed. Moreover, either only the lowest lying states are accounted for in the evaluation of the spectral integrals or, if higher resonances like the π'(1300) in the pseudoscalar channel are included, there is an ambiguity because the coupling constant of this state to the pseudoscalar densities is not known (see [27] for a critical discussion).

- Latice data also prefers a large condensate. In particular, recent high statistic simulations performed by CP-PACS are not compatible with a small condensate. [8] However, these simulations are performed in the quenched approximation, where the extrapolation to the chiral limit is not possible. Although the data looks overhelming, it is not guaranteed that the same behaviour is seen in full simulations. Of course Lattice simulations will be one of the main sources for a better theoretical understanding of the problem in the future.

- The mechanism of SχSB is intimately related to the spectrum of the Euclidean Dirac operator in the background of gauge fields, in a finite box. For instance, the Banks-Casher relation [28]

$$\lim_{m_q \to 0} < 0|\bar{q}q|0 >= -\pi\rho(0)$$

  gives the quark condensate in terms of the spectral density at zero virtuality. In an interesting paper [29] Stern has derived a similar relation for the pion decay constant F_0^2 in the chiral limit. According to this paper, two phases with SχSB can occur: i) the density of states near zero virtuality is large and quarks condense (large condensate scenario). The "mobility" of quarks is suppressed. Microscopically this phase corresponds to weakly localized states. ii) the density of states is too small to generate a quark condensate (zero condensate scenario). However, the mobility of quarks is high which induces a nonzero F_π. Microscopically this phase corresponds to extended delocalized states.

  It wold be very interesting to see a dedicated numerical study of the spectrum of the square of the Euclidean Dirac operator in order to establish in which phase QCD is actually to be found.

8 Conclusions

Chiral Perturbation is the effective theory of low energy QCD. The structure of the theory is dictated by the global chiral symmetry of QCD, and by the assumption that axial-vector charges are spontaneously broken. Predictions depend only on the symmetry properties and are therefore model independent. A vast body of experimental data can be described succesfully.

Two mechanism of spontaneous chiral symmetry breaking have been discussed. In the standard picture the quark condensate is assumed to be large and therefore is the leading order parameter. In generalized ChPT this assumption is relaxed. Nevertheless, SχSB can occur due to other order parameters. If the quark condensate is small, a different chiral counting scheme for the quark masses must be adopted. This leads to predictions of the $\pi\pi$-scattering length which depend strongly on the value of the quark condensate. The issue can be decided in the near future by high precision experiment performed at E865 (Brookhaven), KLOE (DaΦne), and DIRAC (CERN). Another promising avenue is the study of the Euclidean Dirac Operator in a finite box on the lattice.

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