Hidden Symmetry

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1. Introduction

Hidden Symmetry is a phenomenon that occurs when a system that is symmetric has critical points (e.g. potential minima in classical theory, vacua in quantum field theory) that are not symmetric. A simple example that exhibits many of the typical features of hidden symmetry is the following:

Consider four towns at the corners of a square and the problem of joining them by railtracks using the least amount of track. Since the problem has the symmetry of the square, one would at first sight expect the solution to have this symmetry. But to illustrate that it does not, let us consider the possibilities given in Fig.1 given by leaving the angle θ open. It is easy to see that, if we normalize the sides of the square to 2, the full length of track is

$$L(\theta) = \frac{4}{\cos\theta} + 2(1-\tan\theta) \quad \rightarrow \quad L'(\theta) = \frac{4\sin\theta - 2}{(\cos\theta)^2} \tag{1.1}$$

Thus the minimal length is given by

$$\sin\theta_o = \frac{1}{2} \quad \rightarrow \quad \theta_o = \frac{\pi}{6} \tag{1.2}$$

The minimal solution is then as shown in Fig. 2 and does not have the symmetry of the square.

Simple as this example is, it illustrates a number of aspects of SSB also. The first is that the minimum is not unique: although we have chosen the horizontal central line in Fig. 2 we could equally well have chosen the vertical one. The second is that there may be a residual symmetry group because the solution is still invariant under some of the symmetries of the square, left-right and up-down symmetries for example. The third is that the degenerate solutions can be obtained from one another by making one of the symmetry transformations of the square that is *not* a symmetry of a minimum solution. In other words the solutions lie on a group orbit. Thus the symmetry G of the square is not really broken but now manifests itself in a different way, namely as the the residual symmetry group H and the orbit G/H which is just the set cosets of G with respect to H. This is the reason that it is often called a hidden, rather than spontaneously broken, symmetry, though if broken is understood in the sense that G is decomposed into H and G/H this term is equally valid. Finally, the example illustrates the *cost* of preserving the symmetry since to do so there are only two options i.e. either connect the towns diagonally or put in tracks corresponding to *both* solutions, each of which would be more expensive than using the solution. In statistical and field theoretical problems the cost would be in terms of energy.

Although hidden symmetry was not a popular topic in physics until the second half of this century, it has a surprisingly long history. One of the first occurrences was in philosophy where, in the 13th century it appeared in the guise of Buridan's donkey (the donkey that dies of starvation because he is equidistant from two sources of food and cannot choose between them) It occurred later in problems of stability, for example the problem of the vertical pencil. However, the first serious study of the phenomenon was due to Euler, who considered the question as to when bridges, and other objects such as wheels, would buckle under strain. You will hear more about this subject during the Straumann talk. The first major surprise in the field was probably the discovery by Jacobi that axially symmetric rotating fluids under an attractive force such as gravity loose their axial symmetry when the rotational velocity exceeds a certain critical value. This excited much interest at the time because of the possibility of applying the result to the problems of planet formation and binary stars, and is still of interest today for the liquid drop model of nuclear physics. Again you will hear more about this during the Straumann lecture, but let me present a few details as it illustrates another feature of hidden symmetry, namely that it is often the result of a conflict of two opposing forces, each of which is seperately symmetric. What Jacobi considered was a self-gravitating incompressible ellipsoidal rotating fluid with energy

$$E = \frac{1}{2}\omega^2(x_1^2 + x_2^2) + \rho G\Big(C_o - \sum_{k=1}^{k=3} C_k x_k 2\Big) \tag{1.3}$$

where ω is the angular momentum about third axis G is the gravitational constant ρ is the density and C_k are constants that depend on the axes a of the ellipsoid. The conflicting forces here are the gravitational attraction and the centrigugal force engendered by the angular velocity, each of which is axially symmetric. Hydrostic equilibrium requires that at the surface $x_k = a_k$, where the pressure vanishes, the energy should be constant, which gives the condition

$$(\omega^2 - 2c_1)a_1^2 = (\omega^2 - 2c_2)a_2^2 = -2c_3 a_3^2 \tag{1.4}$$

This equation determines the ratios a_k/a_3 of the lengths of the axes and thus the shape. The equations obviously admit the axially symmetric solution $a_1 = a_2$ but they admit non-symmetrical solutions also. What Jacobi discovered was that, for sufficiently large angular momentum ω the non-symmetrical solutions are the stable ones.

Another example that illustrates the conflict between two opposing symmetrical forces is the problem of Magnetic Phase Transitions. To illustrate this let us consider a simplified Model which consists of a one-dimensional lattice of spins ± 1 at each lattice point, average spin (magnetization) χ and free energy

$$F = H + kT\ln(W) = -g\chi^2 + kT\Big[\ln(1-\chi^2) + \chi\ln\Big(\frac{1+\chi}{1-\chi}\Big)\Big] \tag{1.5}$$

Full magnetization (full alignment) means $\chi = \pm 1$ and the tension is between spin-attraction H which tries to magnetize (make $\chi = 1$) and the entropy $kT\ln(W)$ which tries to demagnetize (make $\chi = 0$). Both forces are symmetrical about $\chi = 0$ and so the symmetrical state is the demagnetized one. But since

$$\frac{\partial^2 F}{(\partial\chi)^2} = -2g + \frac{kT}{(1-\chi^2)} \tag{1.6}$$

the state $\chi = 0$ is a minimum of free energy only if $T \geq T_c = 2g/k$. Thus for $T > T_c$ the statistical fluctuations dominate and the model is demagnetized whereas for $T < T_c$ the free-energy minimum is at $\chi \neq 0$ so it is magnetized, reaching full magnetization $\chi = 1$ at zero temperature $T = 0$. The result generalizes to higher dimensions where the symmetry group becomes the rotational group and is continuous.

There is another aspect of hidden symmetry that should be mentioned, namely that it is often a collective phenomenon. This is illustrated by the Salam wineglass example in which a number of people sitting at a circular table and equidistant from the wine-glasses do not know whether to take the one on the right or on the left. But as soon as one person makes a decision the rest are forced to make the same decision. A more serious example is in the case of freezing where the molecules of the liquid not only become crystals but a lattice of aligned crystals. A third example occurs in low-temperature superconductivity, where the Cooper pairs condense to form a condensate that is independent of position. The role of temperature in the last two models is a special case of the fact that hidden symmetry is often associated with one or more order-parameters, whose critical values determine the onset of the symmetry breakdown.

2. Hidden Symmetry in Field Theory

In field theory hidden symmetries usually occur when the scalar field potential is invariant under a symmetry group G but the minimum of the potential is not. The scalar fields $\phi(x)$ are assumed to belong to a linear unitary represntation (not necessarily irreducible) of the group

$$\phi(x) \to U(g)\phi(x) \tag{2.1}$$

and the invariance of the potential means that

$$V(U(g)\phi(x)) = V(\phi(x)) \tag{2.2}$$

For example for a set of real fields $\phi_a(x)$ for $a = 1...n$ the potential $V(\phi(x)) = \phi_a(x)\phi_a(x)$ is SO(n) invariant. As the kinetic terms are always assumed to be invariant under the group the energy

$$H = \int (\partial_t\phi(x))^2 + (\vec{\partial}\phi(x))^2 + V(\phi(x) \tag{2.3}$$

is invariant. Before proceeding to the symmetry breaking it will be convenenient to recall the Noether Theorem. As is well-known every continuous symmetry corresponds to a conserved charge since

$$e^{i\alpha Q} H e^{-i\alpha Q} = H \quad \to \quad [Q, H] = 0 \quad \to \quad e^{itH} Q e^{-itH} = Q \tag{2.4}$$

the first equation of which implies the group invariance of H and the last the conservation of Q. Noether's theorem goes a little farther and in the presence of a local Hamiltonian of the form (2.3) it tells us how to construct the conserved charges Q from conserved currents, namely

$$Q^a = \int d^3x j_o(x) \qquad j^a_\mu(x) = \Big(\partial_\mu \phi(x)\Big)\sigma^a \phi(x) \qquad\qquad \partial_\mu j_\mu(x) = 0 \tag{2.5}$$

where the σ^a are the generators of the transformations (2.1) e.g. Pauli matrices for SU(2). The conservation laws $\partial_\mu j^a_\mu(x) = 0$ is easily established using the equations of motion.

We come now to the symmetry breaking. For this we seek the minimum of the energy. Since both the kinetic and potential terms in the energy are positive, it is clear that the minimum of the energy will occur at a minimum of the potential. But because of the kinetic term it is clear that it will also occur for *constant* field ϕ_o. Thus it is a collective phenomenon in which the field takes the same value at *all* values of x, in analogy to the all the crystals aligning in the same way in the freezing process mentioned earlier.

The question now is whether the point ϕ_o is group invariant. Normally the minimum point is at $\phi_o = 0$ which is obviously group invariant. But there may occur cases where ϕ_o is not zero, in which case it is usually not group invariant. In that case we say that we have a symmetry breakdown or a hidden symmetry. Classic examples are the SO(n) invariant potential, sometimes called the sombrero potential,

$$V(\phi) = \lambda(\phi^2 - c^2)^2 \qquad \phi^2 \equiv \phi_a\phi_a \qquad G = SO(n) \tag{2.6}$$

and the SU(n) invariant potential

$$V(\phi) = \lambda(|\phi|^2 - c^2)^2 \qquad \phi^2 \equiv \phi^\dagger_a\phi_a \qquad G = SU(n) \tag{2.7}$$

sometimes called the sombrero potentials. For these the minimum is obviously some fixed point on the sphere $\phi_0^2 = c^2$ or $\phi^\dagger\phi = c^2$ and therefore cannot be group invariant.

One might ask why we have only considered scalar fields here. The answer is that if fields of any other other spin were constant without being zero they would cause a breakdown of Lorentz invariance and, as far as we know, this does not happen in nature. However, for dimensional reduction other spin fields might well be relevant since in that case the Lorentz symmetry of the higher-dimensional space is broken.

Concept of Little Group

Of course, it may happen that, although the point ϕ_o is not group-invariant there exists some subset of transformations with respect to which it is invariant. These necessarily form a subgroup H,defined by

$$U(h)\phi^o = \phi^o \qquad \text{for some} \quad h \in H \subset G \tag{2.8}$$

The subgroup H is called variously the

little *stability* *isotropy* and *residual symmetry*

group. For the sombrero potential mentioned above the little group H is just the SO(n-1) group of transformations around the axis defined by ϕ_o.

In the broken case any point on the orbit $U(g)\phi^o$ i.e in coset space G/H is a minimum point of the potential, all points on the orbit having the same little group H. Thus as mentioned earlier, the hidden symmetry is characterized by the fact that the minimum of the potential is degenerate. In fact, for hidden symmetries only the orbit is physically meaningful, the choice of point on the orbit being merely a choice of basis.

3. The Role of Dimensions: Energy Cost

If the minimization were with the respect to the potential alone the change from one minimal value ϕ_o of the field to another would cost no energy. But we recall that the quantity to be minimized is not the potential but the *energy*. The presence of the kinetic term not only forces the field to be a constant at the minimum but means that, in general, it will cost energy to move from one constant value of ϕ_o to another. More precisely, if the field moves from the constant configuration ϕ_o to the constant configuration ϕ'_o in time T the amount of energy needed per unit time is

$$E = \frac{1}{T}\int_{\phi_o}^{\phi'_o} dtd^3x\Big[(\partial_t\phi)^2 + (\vec{\partial}\phi)^2 + V(\phi(x))\Big] > 0 \tag{3.1}$$

Since this is proportional to the total volume Ω and inversely proportional to T^2 it is clear that it is strongly dependent on the dimensions. This can also be seen explicitly if we neglect the potential altogether and make the rough estimate

$$E = (\phi'_o - \phi_o)^2\Omega/2T^2 = (\phi'_o - \phi_o)^2 L^d/2T^2 \tag{3.2}$$

where d is the dimension. Let us now consider the different dimensions.

For $d \geq 3$ an analysis of (1) shows that it is not possible to proceed from one constant configuration to another without using an infinite amount of energy. Thus the energy constitutes an insurmountable barrier that prevents the configuration ϕ_o from proceeding to any other value ϕ'_o. *This corresponds to a genuine spontaneous symmetry breakdown.*

The other extreme is the quantum mechanical case d=0, for which the field is just the coordinate of the particle. It is well-known that because of the Heisenberg uncertainty principle ϕ_o cannot then remain at a fixed value. This is evident in the case of a free particle from the spreading of the wave-packet (for which (3.2) can be used to compute the rate) and from the fact that for a sombrero potential the ground state is one in which the wave-function is symmetrically distributed around the minimal orbit. Thus for $d = 0$ the kinetic energy does not constitute a barrier to the spreading of ϕ_o around the minimal orbit and there is no spontaneous symmetry breakdown.

The case $d = 1$ turns out to be same as the quantum mechanical case, except that in general the spread around the minimal orbit proceeds at a slower rate. The case $d = 2$ is a critical case, which requires a more detailed analysis. It turns out that in local field theory there is no spontaneous symmetry breakdown. On the other hand it is known that long-range correlations in statistical mechanics produce a spontaneous symmetry breakdown. So it is a borderline case.

Thus, within the context of field theory, we may say that that for $d = 0, 1, 2$ dimensions there is no spontaneous symmetry breaking but for $d \geq 3$ there is.

4. Scalar and Fermion Masses: the Goldstone Theorem

In a spontaneously broken Lagrangian the scalar potential takes the form

$$U(\theta) = V(\theta + \phi^o) - V(\phi^o) = M_{ik}\theta_i\theta_k + O(\theta^3) + O(\theta^4) \tag{4.1}$$

where

$$M_{ik} = \Big(\frac{\partial^2 V}{\partial_{\phi_i}\partial_{\phi_k}}\Big)_{\phi=0} \tag{4.2}$$

is the scalar mass-matrix. Note that if $V(\phi)$ is renormalizable (quartic) then so is $U(\theta)$. Thus the spontaneous breakdown preserves renormalizability. The mass-matrix M_{ik} must be positive since it is computed at a *minimum* point. But it is not necessarily positive *definite*. In fact, according to the famous *Goldstone Theorem* it has at least dim (G/H) zero eigenvalues. A very simple proof of Goldstone's theorem can be given using the fact that it is actually a special case of the Noether theorem.

We have from the general Noether theorem

$$\partial_\mu j^a_\mu = 0 \qquad j^a_\mu = \Big(\partial_\mu \phi(x)\Big)\sigma^a\phi(x) \tag{4.3}$$

where the σ^a are the generators of the full symmetry group G. Now suppose there is a spontaneous breakdown so that $\phi(x) \to \phi_o + \theta(x)$. Substituting this formula in (3) we obtain

$$(\partial_\mu)^2(\phi^o\sigma_a, \theta(x)) = O(2) \tag{4.4}$$

where $O(2)$ means terms of order at least two in the fluctuating fields i.e. the fields $\theta(x)$ and higher spin fields. Since a field equation for a scalar field $\eta(x)$ of mass m takes the form

$$\big(\partial^2_\mu + m^2\big)\eta(x) = O(2) \tag{4.5}$$

we see from (4) that the fields $(\theta(x), \sigma_a\phi^o)$ must be massless. It remains only to count the number of these fields. Since, by definition, the vector $\sigma^a\phi_o$ is zero if, and only if, the σ_a belongs to the little group H, we see that there are exactly dim G/H such vectors. Thus there are exactly dim G/H fields of the form $(\theta(x), \sigma_a\phi_o)$. Thus there exist at least dim G/H zero-mass scalars.

Fermions

So far we have neglected fermions. If fermions $\psi_r(x)$ are present their contribution to the energy takes the form

$$E_f = \bar{\psi}_r(x)(\vec{\partial} + m_r)\psi_r(x) + G_{rsa}\bar{\psi}_r(x)\psi_s(x)\phi^a(x) \tag{4.6}$$

r and a are internal indices, m_r are (group-invariant) mass-terms and G_{rsa} are Clebsch-Gordan coefficients for the Yukawa term. After a spontaneous symmetry breakdown $\phi(x) \to \phi_o + \theta(x)$ this evidently becomes

$$E_f = \bar{\psi}_r(x)(\vec{\partial} + m_r)\psi_r(x) + \bar{\psi}_r(x)m_{rs}\psi(x) + G_{rsa}\bar{\psi}_r(x)\psi_s(x)\phi^a(x) \tag{4.7}$$

where $m_{rs} = G_{rsa}\phi_o^a$. Thus the effect of the spontaneous breakdown on the fermions is to give them extra mass-terms of the form m_{rs}. In many cases, in particular the case of the electron, muon and tau in the standard electroweak model, there is no original group-invariant mass-term m_r so the mass is completely due to the spontaneous symmetry breakdown.

5. Gauge Field Lagrangian

In the presence of gauge-fields the situation is very different. We first recall that in the presence of gauge-fields the energy takes the form

$$E = \int \vec{E}^2 + \vec{B}^2 + (D_t\phi)^2 + (\vec{D}\phi)^2 + V(\phi) \tag{5.1}$$

where $\vec{E}$ and $\vec{B}$ are the Yang-Mills fields

$$E_i = \partial_o A_i - \partial_i A_o + [A_i, A_o] \quad \text{and} \quad B_i = \epsilon_{ijk}\Big[\partial_j A_k - \partial_k A_j + [A_j, A_k]\Big] \tag{5.2}$$

respectively. Thus the minimum is at the minimum of the potential $\phi_o(x)$ for fields that satisfy

$$\vec{E} = \vec{B} = 0 \qquad D_\mu\phi_o = 0 \tag{5.3}$$

From the first two equations here it follows that A_μ is a pure gauge-field

$$A_\mu = A_\mu^0 = g_o^{-1}(x)\partial_\mu g_o(x) \tag{5.4}$$

and hence from the third equation in (5.3) it follows that

$$\partial_\mu\big(U(g_o^{-1}(x))\phi_o(x)\big) = 0 \quad \to \quad \phi_o(x) = U(g_o(x))\eta_o \tag{5.5}$$

where η_o is constant. In other words, we cannot conclude that $\phi_o(x)$ is constant but only that it is gauge-equivalent to a constant. If the space is topologically trivial, the field A_μ^0 may be gauged to zero so there exists a gauge where $\phi_o(x)$ becomes a constant.

The most important change introduced by the gauge-field is that the Goldstone fields can be gauged to zero and hence are no longer physical fields. More precisely, given the constant vector η_o, we can find a gauge-transformation $\hat{g}(x)$ such that

$$(\hat{\theta}(x), \sigma^a \eta_o) = 0 \quad \text{where} \quad \hat{\theta}(x) = U(\hat{g}(x))\theta(x) \tag{5.6}$$

The proof is a little technical and so is relegated to the Appendix.

6. Gauge-Field Masses

The gauge-field Hamiltonian (5.1) contains no mass-terms for the gauge-fields because such mass-terms are of the form $mA_\mu A_\mu$ and thus are not gauge-invariant. Since the some of the physical gauge-fields, namely the $W^\pm$ and Z^0 fields, and even the electromagnetic field in the superconducting phase, are known experimentally to be massive this constitutes a serious problem. A direct insertion of mass-terms into the energy (or Lagrangian) would destroy the gauge-invariance, and (hence) the renormalizability of the theory, so a more subtle mechanism needed. One of the main uses of hidden symmetry is to provide such a mechanism.

For simplicity of notation we consider the Lagrangian

$$\frac{1}{4}\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + \frac{1}{2}(D_\mu\phi, D_\mu\phi) \tag{6.1}$$

rather than the Hamiltonian, omitting the fermion and potential terms because they play no role in the gauge-field mass mechanism. suppose now that there is a spontaneous symmetry breakdown. Then, on going to the gauge where $\phi_o(x)$ is equal to the constant η_o we have $\phi(x) = \eta_o + \theta(x)$ and the Lagrangian becomes

$$\frac{1}{4}\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + \frac{1}{2}(A_\mu\eta_o, A_\mu\eta_o) + (A_\mu\eta_o, D_\mu\theta(x)) + \frac{1}{2}(D_\mu\theta(x), D_\mu\theta(x)) \tag{6.2}$$

This shows that the gauge-fields acquire the mass-term

$$\frac{1}{2}(A_\mu\eta_o, A_\mu\eta_o) = M_{ab}A^a_\mu A^b_\mu \quad \text{where} \quad M_{ab} = \frac{1}{2}(\sigma_a\eta_o, \sigma_b\eta_o) \tag{6.3}$$

the σ^a's being the generators of the CUR to which the ϕ belong. One might ask how this mass-term differs from an ad hoc one. The answer is that mass-matrix M_{ab} takes a very special form and is accompanied by the cross-terms $(A_\mu\eta_o, D_\mu\theta)$. Since it has a formally gauge-invariant origin this combination is enough to ensure the (formal) gauge-invariance, and hence the renormalizibilty, of the theory. An important point to note from (6.3) is that gauge-fields remain massless if, and only if, they belong to the residual symmetry group H. Thus after the breakdown there are precisely dim G/H massive gauge-fields.

7. Higgs Mechanism

Although spontaneous symmetry breakdown solves the problem of the gauge-field masses, it appears to introduce some further problems. The first is the question of the the Goldstone fields. These are long-range and thus, if they exist, should be seen experimentally. We have seen that there exists a gauge in which they are zero but this only changes the question to whether their non-existence is a gauge dependent statement. (Indeed it was precisely this question for superconductivity that lead to the original Higgs investigation). A second problem is that, because of the indefiniteness of the Minkowski metric $\eta_{\mu\nu}$, the usual canonical commutation relations for the gauge-fields

$$[A_\mu(x), A_\nu(y)] = \eta_{\mu\nu} D(x-y) \tag{7.1}$$

where D the usual light-cone function, produces an indefinite-metric Hilbert space, and hence negative probabilities. In quantum electrodynamics this problem is solved by the Gupta-Bleuler mechanism, through which the contribution from the negative-norm component $\partial_\mu A_\mu$ is cancelled by the contribution from the (non-physical) positive-norm component $\vec{\partial} \cdot \vec{A}$. But this mechanism cannot work for massive gauge-fields, because for them the component $\vec{\partial} \cdot \vec{A}$ is physical. (Recall that in the rest-frame a massive-gauge-field has three directions of polarization.) A third problem is the one mentioned in the Straumann lecture, namely that the spontaneous breakdown of the gauge symmetry seems to contradict the Elitzur theorem, which, in two-dimensional lattice gauge-theory at least, states that there is no breakdown of local symmetry.

Resolution of the Problems

It took physicists some time to realize that there exists a modified version of the Gupta-Bleuler mechanism, called the *Higg's mechanism* through which the above problems are solved. What happens can be seen by using a slightly more sophisticated version of the previous computation. We first note that the argument of the Appendix actually shows that, given *any* constant field η_o there exists a gauge transformation $U(\hat{g}(x))$ that converts the Higgs field $\phi(x)$ to one that is orthogonal to the directions $\sigma_a \eta_o$. Thus the Higgs field $\phi(x)$ has the decomposition

$$\phi(x) = U(\hat{g}(x))\hat{\phi}(x) \quad \text{where} \quad <\hat{\phi}(x), \eta_o>= 0 \tag{7.2}$$

For example for the U(1) gauge group there exists a gauge transformation $e^{i\alpha(x)}$ that converts the complex field $\phi(x)$ to a real field $\rho(x)$. On the other hand the gauge group $G(x)$ may be decomposed into a rigid, x-independent group G_o and a local gauge group $\Gamma(x)$ defined as the group whose elements $\gamma(x)$ become unity at large distances, $\gamma(x) \to 1$ as $x \to \infty$. The gauge group $G(x)$ is actually the semi-direct product of these two groups with $\Gamma(x)$ as the invariant subgroup. The action of the local group $\Gamma(x)$ on $\phi(x)$ may then be absorbed by the $U(\hat{g}(x))$ component of the Higgs field according to

$$U(\hat{g}(x)) \to U(\gamma(x)\hat{g}(x)) \tag{7.3}$$

while $\hat{\phi}(x)$ remains invariant. Meanwhile the action of the rigid gauge-group G_o on $\phi(x)$ is

$$U(\hat{g}(x))\hat{\phi}(x) \to U(h\hat{g}(x)h^{-1})U(h)\hat{\phi}(x) \tag{7.4}$$

The point of this decomposition is that $\hat{\phi}(x)$ is then a scalar with respect to the local gauge-group and has no covariant derivative. Accordingly

$$D_\mu \phi(x) = \Big(D_\mu U(\hat{g}(x))\Big)\hat{\phi}(x) + \partial_\mu \hat{\phi}(x) \tag{7.5}$$

which, if we define the field

$$B_\mu(x) = A_\mu(x) - U^{-1}(\hat{g}(x))\partial U(\hat{g}(x)) \tag{7.6}$$

simplifies to

$$D_\mu \phi(x) = B_\mu \hat{\phi}(x) + \partial_\mu \hat{\phi}(x) \qquad B_\mu(x) \equiv \sigma_a B_\mu^a(x) \tag{7.7}$$

Since the cross-terms in the inner-product vanish we then have

$$\big(D_\mu \phi(x)\big)^2 = \big(B_\mu \hat{\phi}(x)\big)^2 + \big(\partial_\mu \hat{\phi}(x)\big)^2 \tag{7.8}$$

Since $F_{\mu\nu}$ is covariant with respect to the change of field the Lagrangian may be written as

$$\mathrm{tr}\Big(F_{\mu\nu}F^{\mu\nu}(B)\Big) + (\partial_\mu \hat{\phi})^2 + (B_\mu \hat{\phi})^2 + V(\hat{\phi}) \tag{7.9}$$

If there is a spontaneous symmetry breakdown we obtain simply $\hat{\phi}(x) \to \eta_o + \hat{\theta}(x)$, in which case the Lagrangian becomes

$$\mathrm{tr}\Big(F_{\mu\nu}F^{\mu\nu}(B)\Big) + (\partial\hat{\theta}(x))^2 + (B_\mu \eta_o)^2 + 2(B_\mu \eta_o, B_\mu \hat{\theta}(x)) + (B_\mu \hat{\theta}(x))^2 + V(\hat{\phi}(x)) \tag{7.10}$$

The important point is that the fields $B_\mu(x)$ and $\hat{\phi}(x)$ are, by construction, locally gauge invariant. Then, since $(\hat{\phi}(x), \sigma_a \eta_o) = 0$ by definition and $(\eta, \sigma_a \eta_o) = 0$ by the antisymmetry of the group generators, we have $(\hat{\theta}(x), \eta_o) = 0$ and there are no Goldstone fields. Furthermore this is now a gauge-invariant stement. Second, whether or not the symmetry is broken, there is a term in the Lagrangian proportional to B_μ^2 and this means that the timelike component of B_μ is not an independent field. (Recall that classically in this circumstance one of the field equations will be $\partial_\mu(\eta^2 B_\mu) = 0$). Thus there is no field which produces negative probabilities. Finally, since even the spontaneously broken Lagrangian (7.10) is locally gauge-invariant, there is no breakdown of local gauge-invariance, in agreement with the Elitzur theorem. What is actually broken is the *rigid* gauge symmetry.

Appendix

To establish (5.6) and (7.2) consider the quantities

$$< \phi_o, U(g(x))\phi(x) > \equiv < \phi_o, U(e^{\alpha(x)\cdot\sigma})\phi(x) > \tag{A1}$$

As a function of the group parameters $\alpha(x)$ for fixed x, this is a continuous function on a compact space. Hence it has extremal points, $\hat{\alpha}(x)$, which may, of course, de different for each x. Since

$$\frac{\partial}{\partial \alpha_a(x)} U(e^{\alpha(x)\cdot\sigma}) = U(e^{\alpha(x)\cdot\sigma})\sigma_b F_a^b(\alpha(x)) \tag{A2}$$

where $F_a^b(\alpha)$ are the structure functions of the group, we have, at the extremal points,

$$\frac{\partial}{\partial \hat{\alpha}_a(x)} < U(e^{\alpha(x)\cdot\sigma})\sigma_a\phi_o, \phi(x) >= F_a^b(\hat{\alpha}(x)) < \sigma_b\phi_o, U(e^{\alpha(x)\cdot\sigma})\phi(x) >= 0 \qquad (A3)$$

The precise form of the structure functions is not important for our purposes. The important point is that, when regarded as matrices, they are invertible. Hence (A3) implies that

$$< \sigma_b\phi_o, U(e^{\alpha(x)\cdot\sigma})\phi(x) >= 0 \qquad (A4)$$

But this is just the statement that the gauge-transformed fields

$$\hat{\phi}(x) = U(e^{\alpha(x)\cdot\sigma})\phi(x) \qquad (A5)$$

are orthogonal to the Goldstone directions.

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