π^0: lightest hadron

Karol Kampf

IPNP, Charles University, Czech Republic

proud ex-member of PSI theory group

PSI, 16/2/12

Outline:

- basic properties
- decay modes: $\pi^0 \to \gamma\gamma$
- LL
- outlook

π^0's CV

- conception: Yukawa '35, Kemmer '38
- long birth: Lewis, Oppenheimer, Wouthuysen '48,
  Carlson, Hooper, King '50,
  Bjorklund, Crandall, Moyer, York '50 "The existence of a neutral meson is clearly not required at the present stage of the experiments, but is the only one of the above five hypotheses which seems to fit the experimental data."
  Steinberger, Panofsky, Steller '50: "It is clear from these properties that the gamma-rays are the decay products of neutral mesons."
  Ekspong'97: "It was generally felt that the neutral pion marked the end for particle searches."
- two siblings: π^+, π^-, born: Lattes, Muirhead, Occhialini, Powell, '47

π^0's properties

$$I^G\left(J^{PC}\right) = 1^-\left(0^{-+}\right)$$

- mass $m = 134.9766(6)$ MeV
- $m_{\pi^\pm} - m_{\pi^0} = 4.5936(5)$ MeV
- mean life $\tau = (8.4 \pm 0.6) \times 10^{-17}$ s

Why $m_{\pi^0} < m_{\pi^\pm}$ but $m_{K^0} > m_{K^\pm}$?

ChPT+QED: $[\hat{m} = (m_d + m_u)/2]$

$$M^2_{\pi^0} = \qquad 2\hat{m}B_0$$

$$M^2_{\pi^\pm} = 2e^2\frac{C}{F_0^2} + 2\hat{m}B_0$$

+Dashen theorem:

$$M^2_{K^0} = \qquad (m_s + m_d)B_0$$

$$M^2_{K^\pm} = 2e^2\frac{C}{F_0^2} + (m_s + m_u)B_0$$

i.e. depends on isospin breaking, e.g. $R = \frac{m_s - \hat{m}}{m_d - m_u}$
for physical masses: $R = 44$ [note: study of $\eta \to 3\pi$ [K,Knecht,Novotny,Zdrahal'11] $R = 37.7$]

PDG 2008 about π^0, (see new edition for small changes)

π^0 DECAY MODES

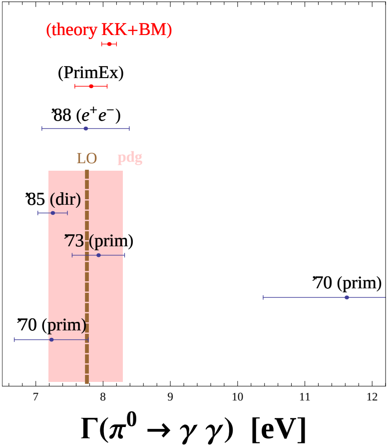
For decay limits to particles which are not established, see the appropriate Search sections (A^0 (axion) and Other Light Boson (X^0) Searches, etc.).

	Mode		Fraction (Γ_i/Γ)	Scale factor/ Confidence level
Γ_1	2γ		(98.798 ± 0.032) %	S=1.1
Γ_2	$e^+e^-\gamma$		(1.198 ± 0.032) %	S=1.1
Γ_3	γ positronium		$(1.82\pm0.29)\times10^{-9}$	
Γ_4	$e^+e^+e^-e^-$		$(3.14\pm0.30)\times10^{-5}$	
Γ_5	e^+e^-		$(6.46\pm0.33)\times10^{-8}$	
Γ_6	4γ		$< 2\times10^{-8}$	CL=90%
Γ_7	$\nu\overline{\nu}$	[a]	$< 2.7\times10^{-7}$	CL=90%
Γ_8	$\nu_e\overline{\nu}_e$		$< 1.7\times10^{-6}$	CL=90%
Γ_9	$\nu_\mu\overline{\nu}_\mu$		$< 1.6\times10^{-6}$	CL=90%
Γ_{10}	$\nu_\tau\overline{\nu}_\tau$		$< 2.1\times10^{-6}$	CL=90%
Γ_{11}	$\gamma\nu\overline{\nu}$		$< 6\times10^{-4}$	CL=90%

Charge conjugation (*C*) or Lepton Family number (*LF*) violating modes

	Mode		Fraction (Γ_i/Γ)	Confidence level
Γ_{12}	3γ	*C*	$< 3.1\times10^{-8}$	CL=90%
Γ_{13}	μ^+e^-	*LF*	$< 3.8\times10^{-10}$	CL=90%
Γ_{14}	μ^-e^+	*LF*	$< 3.4\times10^{-9}$	CL=90%
Γ_{15}	$\mu^+e^- + \mu^-e^+$	*LF*	$< 1.72\times10^{-8}$	CL=90%

π^0 life time



π^0 mean life, PDG history:

1985 $(8.4 \pm 0.6) \times 10^{-17}$ s
...
2009 $(8.4 \pm 0.6) \times 10^{-17}$ s
2010 $(8.4 \pm 0.5) \times 10^{-17}$ s
2011 $(8.4 \pm 0.4) \times 10^{-17}$ s

this year?:
2012 $(8.35 \pm 0.31) \times 10^{-17}$ s

theory: [KK,Moussallam] $(8.04 \pm 0.11) \times 10^{-17}$ s

EFT $\to$ ChPT

EFT

- separated degrees of freedom (simplification)
- building the most general Lagrangian
- ordering principle (powercounting)

example: ChPT

- goldstone bosons (spontaneous symmetry breakdown of chiral symmetry)
- Lagrangian up to NNLO
- dimensional counting

ChPT: effective theory of low-energy QCD

[Coleman, WZ, Callan '69]: nonlinear realization of SSB

$$u(\phi) \to g_R u h^\dagger = h u g_L^\dagger$$

$$u = \exp\left(\frac{i\phi}{F_0 2}\right), \quad \frac{\phi}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & \frac{-2}{\sqrt{6}}\eta \end{pmatrix}$$

$$u_\mu = i(u^\dagger \partial_\mu u - \partial_\mu u u^\dagger - i u^\dagger r_\mu u + i u l_\mu u^\dagger), \quad l_\mu(r_\mu) = v_\mu - (+) a_\mu, \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \quad \chi = 2B_0(s + ip)$$

[Gasser, Leutwyler '84]

$$\mathcal{L}_\chi = \mathcal{L}_\chi^{(2)} + \mathcal{L}_\chi^{(4)} + \mathcal{L}_\chi^{(6)}$$

ChPT: effective theory of low-energy QCD

[Coleman, WZ, Callan '69]: nonlinear realization of SSB

$$u(\phi) \to g_R u h^\dagger = h u g_L^\dagger$$

$$u = \exp\left(\frac{i\phi}{F_0 2}\right), \quad \frac{\phi}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & \frac{-2}{\sqrt{6}}\eta \end{pmatrix}$$

$$u_\mu = i(u^\dagger \partial_\mu u - \partial_\mu u u^\dagger - i u^\dagger r_\mu u + i u l_\mu u^\dagger), \quad l_\mu(r_\mu) = v_\mu - (+) a_\mu, \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \quad \chi = 2B_0 (s + ip)$$

[Gasser, Leutwyler '84]

$$\mathcal{L}_\chi = \mathcal{L}_\chi^{(2)} + \mathcal{L}_\chi^{(4)} + \mathcal{L}_\chi^{(6)}$$

$$\mathcal{L}_\chi^{(2)} = \frac{F_0^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle$$

ChPT: effective theory of low-energy QCD

[Coleman, WZ, Callan '69]: nonlinear realization of SSB

$$u(\phi) \to g_R u h^\dagger = h u g_L^\dagger$$

$$u = \exp\left(\frac{i\phi}{F_0 2}\right), \quad \frac{\phi}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & \frac{-2}{\sqrt{6}}\eta \end{pmatrix}$$

$$u_\mu = i(u^\dagger \partial_\mu u - \partial_\mu u u^\dagger - i u^\dagger r_\mu u + i u l_\mu u^\dagger), \quad l_\mu(r_\mu) = v_\mu - (+) a_\mu, \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \quad \chi = 2B_0(s + ip)$$

[Gasser, Leutwyler '84]

$$\mathcal{L}_\chi = \mathcal{L}_\chi^{(2)} + \mathcal{L}_\chi^{(4)} + \mathcal{L}_\chi^{(6)}$$

$$\begin{aligned} \mathcal{L}_\chi^{(4)} = {} & L_1 \langle u_\mu u^\mu \rangle^2 + L_2 \langle u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle + L_3 \langle u^\mu u_\mu u^\nu u_\nu \rangle + L_4 \langle u^\mu u_\mu \rangle \langle \chi_+ \rangle \\ & + L_5 \langle u^\mu u_\mu \chi_+ \rangle + L_6 \langle \chi_+ \rangle^2 + L_7 \langle \chi_- \rangle^2 + L_8 \langle \chi_+ \chi_- \rangle \\ & - i L_9 \langle F^R_{\mu\nu} u u^\mu u^\nu u^\dagger + F^L_{\mu\nu} u^\dagger u^\mu u^\nu u \rangle + L_{10} \langle F^R_{\mu\nu} U F^{L\mu\nu} U^\dagger \rangle \\ & + H_1 \langle F^R_{\mu\nu} F^{\mu\nu R} + F^L_{\mu\nu} F^{\mu\nu L} \rangle + H_2 \langle \chi_+^2 - \chi_-^2 \rangle / 4 \end{aligned}$$

ChPT: effective theory of low-energy QCD

[Coleman, WZ, Callan '69]: nonlinear realization of SSB

$$u(\phi) \to g_R u h^\dagger = h u g_L^\dagger$$

$$u = \exp\left(\frac{i\phi}{F_0 2}\right), \quad \frac{\phi}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & \frac{-2}{\sqrt{6}}\eta \end{pmatrix}$$

$$u_\mu = i(u^\dagger \partial_\mu u - \partial_\mu u u^\dagger - i u^\dagger r_\mu u + i u l_\mu u^\dagger), \quad l_\mu(r_\mu) = v_\mu - (+)a_\mu, \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \quad \chi = 2B_0(s + ip)$$

[Gasser, Leutwyler '84]

$$\mathcal{L}_\chi = \mathcal{L}_\chi^{(2)} + \mathcal{L}_\chi^{(4)} + \mathcal{L}_\chi^{(6)}$$

$$\mathcal{L}_\chi^{(6)} = C_1 \langle u \cdot u h_{\mu\nu} h^{\mu\nu} \rangle + \ldots \text{ (together 94 terms!)}$$

where $f_\pm^{\mu\nu} = u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u, \quad \nabla_\lambda f_\pm^{\mu\nu}, \quad h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu$

$$\chi_{\pm\mu} = u^\dagger D_\mu \chi u^\dagger \pm u D_\mu \chi^\dagger u = \nabla_\mu \chi_\pm - \frac{i}{2}\{\chi_\mp, u_\mu\}$$

[Bijnens, Colangelo, Ecker '99]

Odd sector

So far, the effective Lagrangian has a larger symmetry than QCD.

$\phi \leftrightarrow -\phi$ in this Lagrangian: only even number of GB

odd intrinsic parity is however not symmetry of original QCD

from phenomenology we know it exists: $K^+K^- \to 3\pi$

$\Rightarrow$ symmetry pattern of QCD must be studied more carefully

Odd sector

- first we need to add EM interaction:
  $\partial_\mu U \to D_\mu U = \partial_\mu U + i[U, v_\mu], \qquad v_\mu \sim Q A_\mu$
- and add by hands monomial to $\mathcal{L}$:

$$U F_{\mu\nu} \tilde{F}^{\mu\nu}$$

  U can be transformed out: we have to add (at least two) derivatives on U – vanishes in chiral limit (Sutherland theorem)

way out: anomaly, in fact two anomalies (non-trivial for $i = 0, 3, 8$, or for π^0, η, η' states):

$$\partial^\mu A^i_\mu = N_f \delta^{i0} \frac{\alpha_s}{4\pi} G^{\mu\nu a} \tilde{G}^a_{\mu\nu} + a^i \frac{\alpha}{4\pi} F^{\mu\nu} \tilde{F}_{\mu\nu}$$

incorporated to the action by Wess, Zumino and Witten (WZW)
two-flavour case: [Kaiser'01]

Anomalies in SM: the good, the bad and the ugly

An "anomaly": the symmetry of the classical action violated by quantum correction.

the good: Sutherland theorem in contradiction with experiment for $\pi^0 \to \gamma\gamma$. Precluded by chiral anomaly.

the bad: in EW lepton sector we have an anomaly, the theory is internally inconsistent; solution "BIM" mechanism: lepton and quark must be treated together

the ugly: dubbed $U(1)_A$ problem by Weinberg: we have 8 not 9 goldstone bosons. η' is not protected by $U(1)_A$ as this is not a good symmetry (is affected by the anomaly): $\partial_\mu J_5^\mu \sim G_a^{\mu\nu}\tilde{G}_{a\mu\nu}$
however, this is a total derivative, but it contributes ['t Hooft '76]
it begets another problem: we should add to the Lagrangian $\theta G\tilde{G}$.
But θ (including chiral quark mass phase) is small, why? $\to$ strong CP problem

Calculation within 2-flavour ChPT: Anomaly

Wess-Zumino construction ['71]
The form of anomaly is determined by 'gauging' external fields, where v and a are gauge fields of

$$L = R = 1 + i\alpha, \quad \text{and} \quad L^\dagger = R = 1 + i\beta$$

$$\delta S\{v,a,s,p,\theta\} = -\frac{N_C}{16\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\langle\hat\beta(\hat v_{\mu\nu} + i[\hat a_\mu, \hat a_\nu])\rangle\langle v_{\rho\sigma}\rangle$$

$$\mathcal{L}_{\text{WZW}} = -\frac{N_C}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\big\{\langle U^\dagger \hat r_\mu U \hat l_\nu - \hat r_\mu \hat l_\nu + i\Sigma_\mu(U^\dagger \hat r_\nu U + \hat l_\nu)\rangle\langle v_{\rho\sigma}\rangle + \tfrac{2}{3}\langle\Sigma_\mu\Sigma_\nu\Sigma_\rho\rangle\langle v_\sigma\rangle\big\}$$

with $\Sigma_\mu = U^\dagger\partial_\mu U, \quad \hat r_\mu = \hat v_\mu + \hat a_\mu, \quad \hat l_\mu = \hat v_\mu - \hat a_\mu.$

Calculation within 2-flavour ChPT

NLO odd-intrinsic Lagrangian is given by [Bijnens, Girlanda, Talavera '02]

$$\mathcal{L}_6^W = \sum_{i=1}^{13} c_i^W o_i^W \,, \qquad c_i^W = c_i^{Wr} + \eta_i (c\mu)^{d-4}\Lambda \,,$$

monomial (o_i^W)	i 2-flavour	$384\pi^2 F^2 \eta_i$
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ [f_{-\mu\nu}, u_\alpha u_\beta]\rangle$	1	0
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_- \{f_{+\mu\nu}, u_\alpha u_\beta\}\rangle$	2	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{+\mu\nu} f_{+\alpha\beta}\rangle$	3	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{-\mu\nu} f_{-\alpha\beta}\rangle$	4	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ [f_{+\mu\nu}, f_{-\alpha\beta}]\rangle$	5	0
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle\chi_- u_\alpha u_\beta\rangle$	6	$-5N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\chi_-\rangle$	7	$4N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\rangle\langle\chi_-\rangle$	8	$-2N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu}\rangle\langle h_{\gamma\nu} u_\alpha u_\beta\rangle$	9	$2N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu}\rangle\langle f_{-\gamma\nu} u_\alpha u_\beta\rangle$	10	$-6N_C$
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\gamma\alpha} h_{\gamma\beta}\rangle$	11	$4N_C$
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\gamma\alpha} f_{-\gamma\beta}\rangle$	12	0
$\epsilon^{\mu\nu\alpha\beta}\langle\nabla_\gamma f_{+\gamma\mu}\rangle\langle f_{+\nu\alpha} u_\beta\rangle$	13	$-4N_C$

n.b. it depends on the form of $\mathcal{L}_4$ [KK, Novotny 02], [Ananth.,Moussallam 02]

$\pi^0 \to \gamma\gamma$: chiral expansion

collaboration with B. Moussallam

- π^0 lightest hadron $\Rightarrow$ primary decay mode $\pi^0 \to \gamma\gamma$
- in chiral limit exact due to QCD axial anomaly:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{m_{\pi^0}^3}{64\pi}\left(\frac{\alpha N_C}{3\pi F_\pi}\right)^2 = 7.73\,eV$$

Correction to the current algebra prediction:

- using [Pagels and Zepeda '72] sum rules in [Kitazawa '85]
- NLO corrections are hidden in $F \to F_{\pi^0}$ and $O(p^6)$ LECs [Donoghue, Holstein, Lin '85] [Bijnens, Bramon, Cornet '88]
- in 3-flavour case we can study π^0, η, η' mixing, resulting to [Goity, Bernstein, Holstein '02]:

$$\Gamma^{\text{NLO}} = 8.1 \pm 0.08\,\text{eV}$$

  in 2-flavour case EM corrections [Ananth., Moussallam '02]:

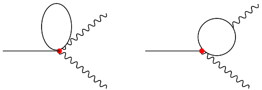
$$\Gamma^{\text{NLO}} = 8.06 \pm 0.02 \pm 0.06\,\text{eV}$$

- Quite recently another study based on dispersion relations, QCD sum rules, using only the value $\Gamma(\eta \to \gamma\gamma)$ gives [Ioffe, Oganesian '07]:

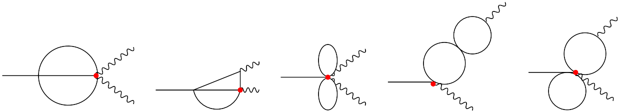
$$\Gamma^{\text{NLO}} = 7.93 \pm 0.11\,\text{eV}$$

$\pi^0 \to \gamma\gamma$ at NNLO in 2 flavour ChPT: technical part

- NLO: a) One-loop diagrams with one vertex from $\mathcal{L}^{WZ}$, b) tree diagrams with one vertex from $\mathcal{L}^{WZ}$ and one vertex from $O(p^4)$ Lagrangian, c) tree diagrams with one vertex from $O(p^6)$ anomalous-parity sector
- $O(p^6)$ anomalous-parity sector from [Bijnens, Girlanda, Talavera '02]
- representation of chiral field: $U = \sigma + i\frac{\tau.\pi}{F}$, $\sigma = \sqrt{1 - \vec{\pi}^2/F^2}$ (no $\gamma 4\pi$ vertex at LO)
- one-loop

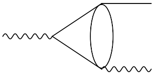


- two-loop



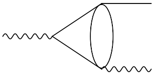
- verification of Z-factor, F_π/F [Bürgi '96], [Bijnens, Colangelo, Ecker, Gasser, Sainio '02]
- double log checked by Weinberg consistency rel. [Colangelo '95]

$\pi^0 \to \gamma\gamma$ technical part: some details on two-loop calculation



$$\mathcal{T} \sim \epsilon^{\mu\nu\rho\sigma} e^1_\alpha\, k^2_\rho\, e^2_\sigma\, p_\lambda$$
$$\times \int \frac{d^d l_1}{i(2\pi)^d} \frac{d^d l_2}{i(2\pi)^d} \frac{l_1^\lambda\, l_1^\mu\, l_2^\alpha\, l_2^\nu}{(l_1^2 - M^2)(l_2^2 - M^2)[(l_2+k_1)^2 - M^2][(l_1+l_2-k_2)^2 - M^2]}$$

$\pi^0 \to \gamma\gamma$ technical part: some details on two-loop calculation



$$\mathcal{T} \sim \epsilon^{\mu\nu\rho\sigma} e^1_\alpha \, k^2_\rho \, e^2_\sigma \, p_\lambda$$

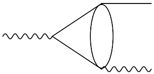
[Bijnens, Colangelo, Ecker, Gasser, Sainio '97]

$$\times \int \frac{d^d l_1}{i(2\pi)^d} \frac{d^d l_2}{i(2\pi)^d} \frac{l_1^\lambda \, l_1^\mu \, l_2^\alpha \, l_2^\nu}{(l_1^2 - M^2)(l_2^2 - M^2)[(l_2+k_1)^2 - M^2][(l_1+l_2-k_2)^2 - M^2]}$$

$$J(t) = \int \frac{d^d l_1}{i(2\pi)^d} \frac{l_1^\lambda \, l_1^\mu}{(l_1^2 - M^2)((l_1+t)^2 - M^2)} \qquad \to \qquad \frac{g_{\lambda\mu}}{4(2w+3)} \int_{4M^2}^\infty \frac{[d\sigma]}{\sigma - t^2}(4M^2 - \sigma)$$

$$\Rightarrow R^{\alpha\nu} = \frac{1}{4(2w+3)} \int \frac{d^d l_2}{i(2\pi)^d} \frac{l_2^\alpha \, l_2^\nu}{(l_2^2 - M^2)((l_2+k_1)^2 - M^2)} \int_{4M^2}^\infty \frac{[d\sigma](4M^2 - \sigma)}{\sigma - t^2}$$

$\pi^0 \to \gamma\gamma$ technical part: some details on two-loop calculation



$$\mathcal{T} \sim \epsilon^{\mu\nu\rho\sigma} e^1_\alpha\, k^2_\rho\, e^2_\sigma\, p_\lambda$$

[Bijnens, Colangelo, Ecker, Gasser, Sainio '97]

$$\times \int \frac{d^d l_1}{i(2\pi)^d} \frac{d^d l_2}{i(2\pi)^d} \frac{l_1^\lambda\, l_1^\mu\, l_2^\alpha\, l_2^\nu}{(l_1^2-M^2)(l_2^2-M^2)[(l_2+k_1)^2-M^2][(l_1+l_2-k_2)^2-M^2]}$$

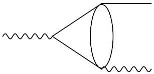
$$J(t) = \int \frac{d^d l_1}{i(2\pi)^d} \frac{l_1^\lambda\, l_1^\mu}{(l_1^2-M^2)((l_1+t)^2-M^2)} \quad \to \quad \frac{g_{\lambda\mu}}{4(2w+3)} \int_{4M^2}^\infty \frac{[d\sigma]}{\sigma - t^2}(4M^2-\sigma)$$

$$\Rightarrow R^{\alpha\nu} = \frac{1}{4(2w+3)} \int \frac{d^d l_2}{i(2\pi)^d} \frac{l_2^\alpha\, l_2^\nu}{(l_2^2-M^2)((l_2+k_1)^2-M^2)} \int_{4M^2}^\infty \frac{[d\sigma](4M^2-\sigma)}{\sigma - t^2}$$

Divergences can be separated by Taylor expansion around $\sigma = \infty$

$$\int_{4M^2}^\infty [d\sigma](4M^2-\sigma)\sigma^l = -\frac{2(2w+3)}{(4\pi)^{2+\omega}}\Gamma(-2-w-l)\frac{\Gamma(-l)}{\Gamma(-2l)}(M^2)^{w+2+l}, \qquad \text{conv. for } l < -2$$

$\pi^0 \to \gamma\gamma$ technical part: some details on two-loop calculation



$$\mathcal{T} \sim \epsilon^{\mu\nu\rho\sigma} e^1_\alpha\, k^2_\rho\, e^2_\sigma\, p_\lambda$$

[Bijnens, Colangelo, Ecker, Gasser, Sainio '97]

$$\times \int \frac{d^d l_1}{i(2\pi)^d} \frac{d^d l_2}{i(2\pi)^d} \frac{l_1^\lambda\, l_1^\mu\, l_2^\alpha\, l_2^\nu}{(l_1^2-M^2)(l_2^2-M^2)[(l_2+k_1)^2-M^2][(l_1+l_2-k_2)^2-M^2]}$$

$$J(t) = \int \frac{d^d l_1}{i(2\pi)^d} \frac{l_1^\lambda\, l_1^\mu}{(l_1^2-M^2)((l_1+t)^2-M^2)} \quad \to \quad \frac{g_{\lambda\mu}}{4(2w+3)} \int_{4M^2}^\infty \frac{[d\sigma]}{\sigma - t^2}(4M^2-\sigma)$$

$$\Rightarrow R^{\alpha\nu} = \frac{1}{4(2w+3)} \int \frac{d^d l_2}{i(2\pi)^d} \frac{l_2^\alpha\, l_2^\nu}{(l_2^2-M^2)((l_2+k_1)^2-M^2)} \int_{4M^2}^\infty \frac{[d\sigma](4M^2-\sigma)}{\sigma - t^2}$$

Divergences can be separated by Taylor expansion around $\sigma = \infty$

$$\int_{4M^2}^\infty [d\sigma](4M^2-\sigma)\sigma^l = -\frac{2(2w+3)}{(4\pi)^{2+\omega}}\Gamma(-2-w-l)\frac{\Gamma(-l)}{\Gamma(-2l)}(M^2)^{w+2+l}, \qquad \text{conv. for } l < -2$$

so far one would obtain:

$$R = \frac{3581}{8064} + \frac{\pi^2}{24} + R_1 + R_2\,, \qquad \text{with} \qquad R_1 = -\frac{1}{84}\int_4^\infty ds \sqrt{\frac{(s-4)^3}{s}}(\log(s) pol_1 + pol_2)$$

and R_2 can be expressed as double integral, where one integral comes from

$$I = \frac{M^6}{4}\int dxdydz \int \frac{d^d l}{i(2\pi)^d} \frac{60\, x^3 y^2 z^3 (1-z)^4\, l^2}{[A_z - x(1-y)B_z - l^2]^6} \quad \text{with} \quad A_z = z\sigma + (1-z)M^2, \quad B_z = z(1-z)M^2$$

n.b. possible expansion in $B_z/A_z \leq 1/9$

$\pi^0 \to \gamma\gamma$ at NNLO, result

$$
\begin{aligned}
A_{NNLO} = \frac{e^2}{F_\pi}\Big\{ &\frac{1}{4\pi^2} \\
&+ \frac{16}{3} m_\pi^2 (-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr}) + \frac{64}{9} B(m_d - m_u)(5c_3^{Wr} + c_7^{Wr} + 2c_8^{Wr}) \\
&- \frac{M^4}{24\pi^2 F^4}\left(\frac{1}{16\pi^2} L_\pi\right)^2 + \frac{M^4}{16\pi^2 F^4} L_\pi \left[\frac{3}{256\pi^4} + \frac{32F^2}{3}\left(2c_2^{Wr} + 4c_3^{Wr} + 2c_6^{Wr} + 4c_7^{Wr} - c_{11}^{Wr}\right)\right] \\
&+ \frac{32M^2 B(m_d - m_u)}{48\pi^2 F^4} L_\pi \left[-6c_2^{Wr} - 11c_3^{Wr} + 6c_4^{Wr} - 12c_5^{Wr} - c_7^{Wr} - 2c_8^{Wr}\right] \\
&+ \frac{M^4}{F^4}\lambda_+ + \frac{M^2 B(m_d - m_u)}{F^4}\lambda_- + \frac{B^2(m_d - m_u)^2}{F^4}\lambda_{--}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_+ &= \frac{1}{\pi^2}\left[-\frac{2}{3} d_+^{Wr}(\mu) - 8c_6^r - \frac{1}{4}(l_4^r)^2 + \frac{1}{512\pi^4}\left(-\frac{983}{288} - \frac{4}{3}\zeta(3) + 3\sqrt{3}\,\mathrm{Cl}_2(\pi/3)\right)\right] \\
&\quad + \frac{16}{3} F^2 \left[8 l_3^r (c_3^{Wr} + c_7^{Wr}) + l_4^r(-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr})\right] \\
\lambda_- &= \frac{64}{9}\left[d_-^{Wr}(\mu) + F^2 l_4^r (5c_3^{Wr} + c_7^{Wr} + 2c_8^{Wr})\right] \\
\lambda_{--} &= d_{--}^{Wr}(\mu) - 128F^2 l_7 (c_3^{Wr} + c_7^{Wr})\ .
\end{aligned}
$$

- 4 LECs in 2 combinations of NLO
- additional 4 LECs in 3 combinations of NNLO

Is it at all possible to make some reliable prediction?

$\pi^0 \to \gamma\gamma$: modified counting

- Use of $SU(3)$ phenomenology via $c_i^{Wr} \leftrightarrow C_i^{Wr}$ connection (based on [Gasser, Haefeli, Ivanov, Schmid '07,'08])

$$c_i^{Wr} = \frac{\alpha_i}{m_s} + \left(\beta_i + \gamma_{ij} C_j^{Wr} + \delta_i \ln \frac{B_0 m_s}{\mu^2}\right) + O(m_s)$$

$\pi^0 \to \gamma\gamma$: modified counting

- Use of $SU(3)$ phenomenology via $c_i^{Wr} \leftrightarrow C_i^{Wr}$ connection (based on [Gasser, Haefeli, Ivanov, Schmid '07,'08])
$$c_i^{Wr} = \frac{\alpha_i}{m_s} + \left(\beta_i + \gamma_{ij} C_j^{Wr} + \delta_i \ln \frac{B_0 m_s}{\mu^2}\right) + O(m_s)$$
- implementation of modified counting

$$m_u, m_d \sim O(p^2) \quad \text{and} \quad m_s \sim O(p)$$

Result:

$$A_{NNLO}^{mod} = \frac{e^2}{F_\pi}\left\{\frac{1}{4\pi^2} - \frac{64}{3} m_\pi^2 C_7^{Wr} + \frac{1}{16\pi^2}\frac{m_d - m_u}{m_s}\left[1 - \frac{3}{2}\frac{m_\pi^2}{16\pi^2 F_\pi^2} L_\pi\right]\right.$$
$$+ 32B(m_d - m_u)\left[\frac{4}{3}C_7^{Wr} + 4C_8^{Wr}\left(1 - 3\frac{m_\pi^2}{16\pi^2 F_\pi^2} L_\pi\right)\right.$$
$$\left.\left. - \frac{1}{16\pi^2 F_\pi^2}\left(3L_7^r + L_8^r - \frac{1}{512\pi^2}(L_K + \frac{2}{3}L_\eta)\right)\right] - \frac{1}{24\pi^2}\left(\frac{m_\pi^2}{16\pi^2 F_\pi^2} L_\pi\right)^2\right\}$$

$\pi \to \gamma\gamma$: Phenomenology

- $F_\pi = 92.22 \pm 0.07$ MeV (using updated value of V_{ud} [Towner, Hardy'08]).
  rem.: if SM violated: $F_\pi \to \hat{F}_\pi$ [Bernard, Oertel, Passemar, Stern '08]

  using quark mass ratio (from lattice), pseudo-scalar meson masses, R from $\eta \to 3\pi$ (cf. [K,Knecht,Novotny,Zdrahal])
- $\frac{m_d - m_u}{m_s} = (2.29 \pm 0.23)\, 10^{-2}$
- $B(m_d - m_u) = (0.32 \pm 0.03)\, M_{\pi^0}^2$
- $3L_7 + L_8^r(\mu) = (0.10 \pm 0.06)\, 10^{-3} \quad (\mu = M_\eta)$ (from pseudo-scalar meson masses formula [Gasser, Leutwyler '85])
- $C_7^W = 0$ (more precisely $C_7^W \ll C_8^W$, motivated by simple resonance saturation)
- $C_8^W = (0.58 \pm 0.2) 10^{-3}$GeV^{-2} (from $\eta \to 2\gamma$)

result $\Gamma_{\pi^0 \to 2\gamma} = (8.09 \pm 0.11)$eV

Too complicated?

Too complicated?

one loop only in the next...

Renormalizable theories

- we calculate e.g. $F(M)$:

$$\begin{aligned} F &= F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + \ldots \\ &= \alpha + \alpha^2 f_1^1 L + \alpha^2 f_0^1 + \alpha^3 f_2^2 L^2 + \alpha^3 f_1^2 L + \alpha^3 f_0^2 + \ldots \end{aligned}$$

- where we have defined $L \equiv \log(\mu/M)$
- renormalization condition $\mu \frac{dF}{d\mu} = 0$
- non-trivial dependence on α

$$\mu \frac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \ldots$$

- β_0 obtained from 1-loop diagrams
- renormalization condition $\Rightarrow$

$$f_1^1 = -\beta_0,\ f_2^2 = \beta_0^2,\ f_3^3 = -\beta_0^3 \quad \Rightarrow \quad F\big|_{LL} = \frac{\alpha}{1 + \alpha\beta_0 L}$$

Non-renormalizable theories

What are the Leading Logarithms (LL)?

- We calculate e.g. $F(M)$:

$$F = F_0 + \underline{F_1^1 L} + F_0^1 + \underline{F_2^2 L^2} + F_1^2 L + F_0^2 + \dots$$

- where we have defined $L \equiv \log(\mu/M)$

Why they are special?

- they are parameter-free
- to all orders from one-loop diagrams only (based on [Weinberg '79], [Büchler, Colangelo'03])

$O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model

$$\mathcal{L}_{n\sigma} = \frac{F^2}{2}\partial_\mu \Phi^T \partial^\mu \Phi + F^2 \chi^T \Phi\,.$$

- explicit + spontaneous symmetry breaking

$$\langle \Phi^T \rangle = (1\ 0 \ldots 0) \qquad \chi^T = (M^2\ 0 \ldots 0)$$

- we have N Goldstone bosons: ϕ
- $N = 3$ equivalent to two-flavour ChPT

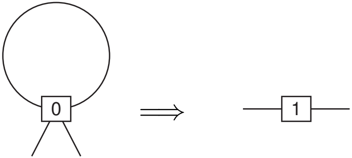
$O(N)$ sigma model: physical mass M_π

LL for the physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- $\mathcal{L}_{n\sigma} \Rightarrow \boxed{0}$
- mass: two-point function
- schematically at LO:

$$\Leftrightarrow \quad M_\pi = M$$

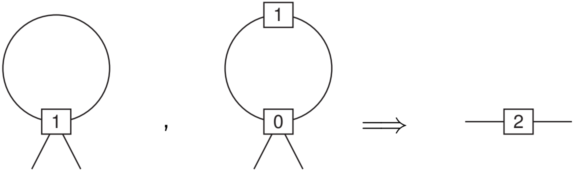
- NLO



$O(N)$ sigma model: physical mass M_π

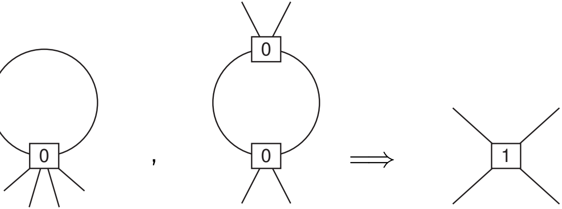
LL for physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- NNLO



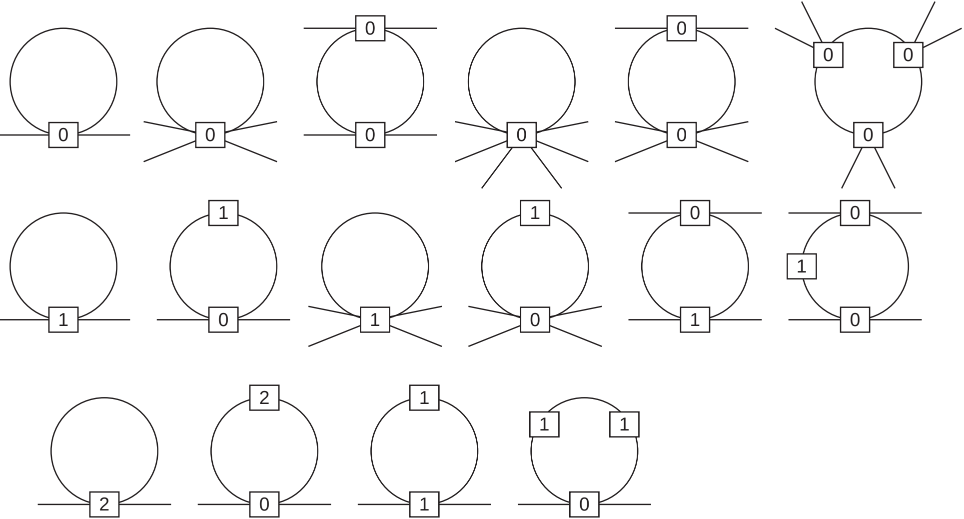
(1)

- we have to calculate first:

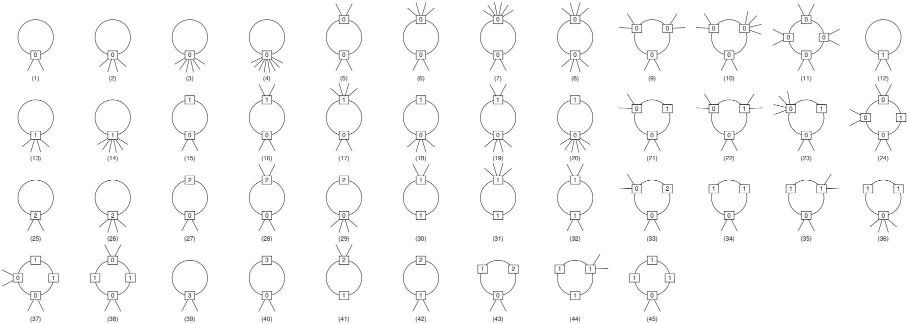


(2)

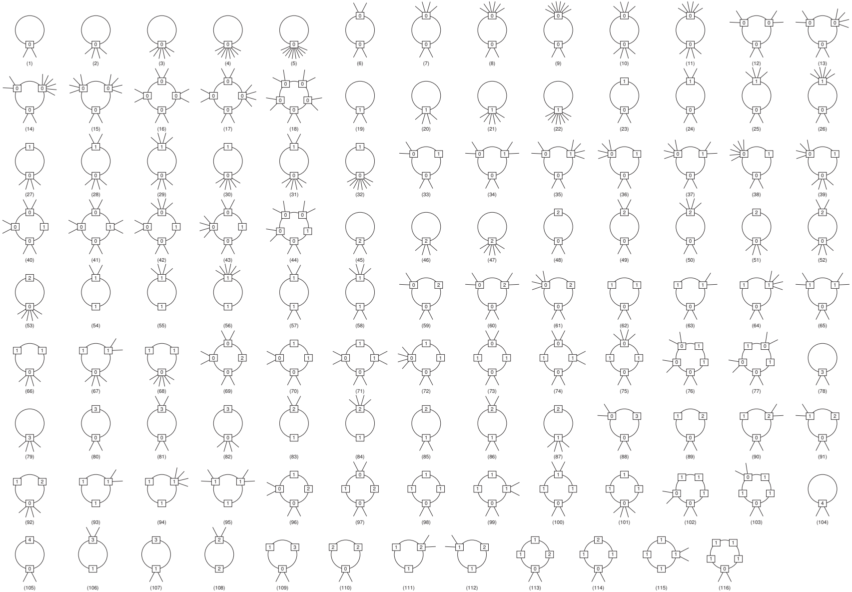
mass up-to 3-loop order



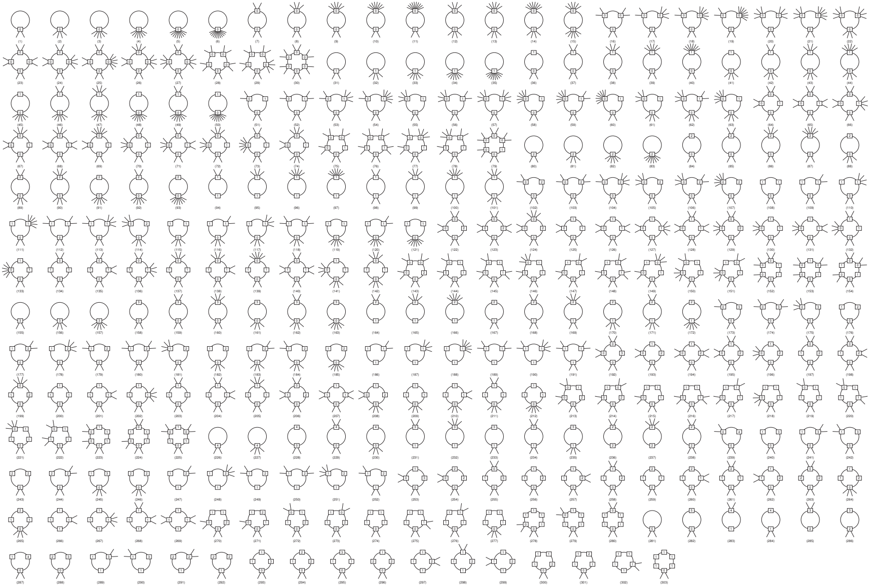
mass up-to 4-loop order



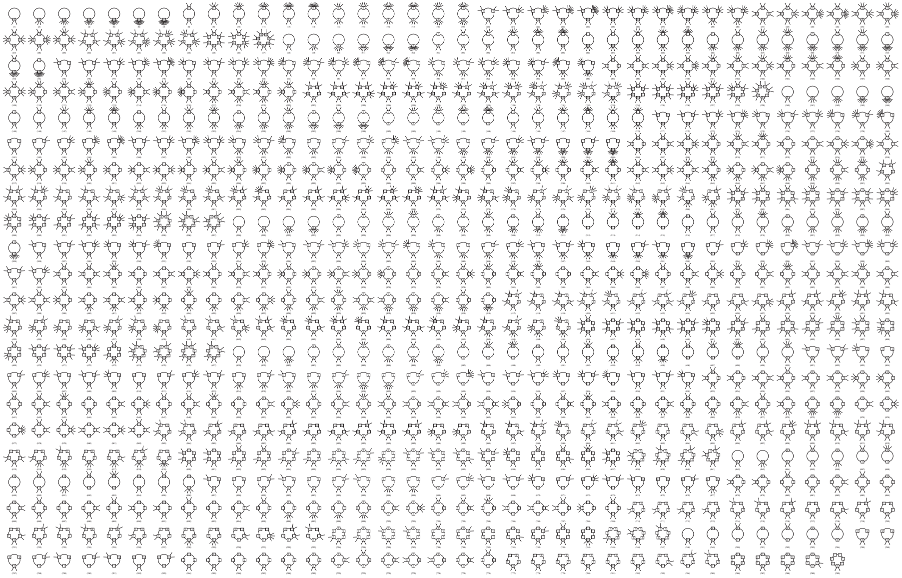
mass up-to 5-loop order



mass up-to 6-loop order



mass up-to 7-loop order



$O(N)$ sigma model: physical mass M_π, results

LL for physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- # of diagrams: $1, 5, 16, 45, 116, 303, 790, \ldots$
- $M_\pi^2 = M^2(1 + a_1 L_M + a_2 L_M^2 + \ldots)$
  $L_M = M^2/(16\pi^2 F^2) \log(\mu^2/M^2)$

i	a_i for $N=3$	a_i for general N
1	$-1/2$	$1 - 1/2\ N$
2	17/8	$7/4 - 7/4\ N + 5/8\ N^2$
3	$-103/24$	$37/12 - 113/24\ N + 15/4\ N^2 - N^3$
4	24367/1152	$839/144 - 1601/144\ N + 695/48\ N^2 - 135/16\ N^3 + 231/128\ N^4$
5	$-8821/144$	$33661/2400 - 1151407/43200\ N + 197587/4320\ N^2 - 12709/300\ N^3$ $+6271/320\ N^4 - 7/2\ N^5$
6	$\frac{1922964667}{6220800}$	$158393809/3888000 - 182792131/2592000\ N + 1046805817/7776000\ N^2$ $-17241967/103680\ N^3 + 70046633/576000\ N^4 - 23775/512\ N^5$ $+7293/1024\ N^6$
7*	$-\frac{1804453729667}{1714608000}$	$1098817478897/8573040000 - 286907006651/1428840000\ N$ $+4533157401977/11430720000\ N^2 - 1986536871797/3429216000\ N^3$ $+436238667943/762048000\ N^4 - 7266210703/21168000\ N^5$ $+99977/896\ N^6 - 15\ N^7$

* preliminary

$O(N)$ sigma model: physical mass M_π, verification

Some cross-check? yes

- different parameterizations, (we have used 5) e.g.

$$\Phi_1 = \begin{pmatrix} \sqrt{1 - \frac{\phi^T\phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}, \qquad \Phi_4 = \begin{pmatrix} \cos\sqrt{\frac{\phi^T\phi}{F^2}} \\ \sin\sqrt{\frac{\phi^T\phi}{F^2}}\frac{\phi}{\sqrt{\phi^T\phi}} \end{pmatrix}$$

- full two-loop result for $N = 3$
  - for $N = 3$ we have $SU(2) \times SU(2)/SU(2)$
  - well known: [Colangelo '95], [Bürgi '96], [Bijnens et al. '97]
- limits
  - massless limit
  - large N limit

$O(N)$ sigma model: physical mass M_π, verification

Some cross-check? yes

- different parameterizations, (we have used 5) e.g.

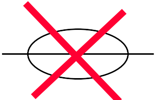
$$\Phi_1 = \begin{pmatrix} \sqrt{1-\frac{\phi^T\phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}, \qquad \Phi_4 = \begin{pmatrix} \cos\sqrt{\frac{\phi^T\phi}{F^2}} \\ \sin\sqrt{\frac{\phi^T\phi}{F^2}}\frac{\phi}{\sqrt{\phi^T\phi}} \end{pmatrix}$$

- full two-loop result for $N=3$
  - for $N=3$ we have $SU(2)\times SU(2)/SU(2)$
  - well known: [Colangelo '95], [Bürgi '96], [Bijnens et al. '97]
- limits
  - massless limit
  - large N limit

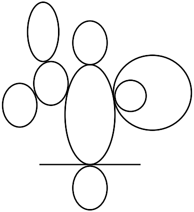
$O(N)$ sigma model: physical mass M_π, large N

[Bijnens, Carloni '09]

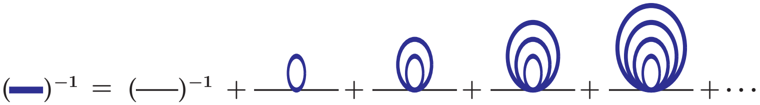
- powercounting: $F \sim \sqrt{N}$, $M^2 \sim 1$
- no lines shared between two loops



- as many closed flavour loops as there are loops
- only tadpoles $\Rightarrow$ cactus diagrams



$O(N)$ sigma model: physical mass M_π, large N



in parametrization 1 (Φ_1) ($Z_p = 1$):

$$p^2 - B_p = p^2 - M^2 - M^2\left(\frac{1}{\sqrt{1+\frac{N}{F^2}A(B_p)}} - 1\right)$$

result (complete)

$$M^2 = M_\pi^2\sqrt{1+\frac{N}{F^2}A(M_\pi^2)}, \quad \text{kde} \quad iA(B) = \int d^d p 1/(p^2 - B)$$

$O(N)$ sigma model

already calculated [Bijnens, Carloni '09,'10], [Bijnens, KK, Lanz '12]

- mass M_π
- pion decay constant F_π
- $\phi\phi$ scattering, F_V, F_S

massless case [Kivel, Polyakov, Vladimirov '08,'09,'10,'10]

- $\phi\phi$ scattering, F_V, F_S
- recursive relations

N.B. knowing F_π enables the expansion both in $L \sim M^2/F^2 \times \log M$
and in $L_\pi \sim M_\pi^2/F_\pi^2 \times \log M_\pi$

$O(N)$ sigma model: vector formfactor

- for $\pi \to \gamma\gamma$ we will need even number of pions and one photon
- effectively hidden in F_V
- definition

$$\langle \phi^a(\mathbf{p}_f)| j^{cd}_{V,\mu} - j^{dc}_{V,\mu} |\phi^b(\mathbf{p}_i)\rangle = \left(\delta^{ac}\delta^{db} - \delta^{ad}\delta^{bc}\right) i(\mathbf{p}_f + \mathbf{p}_i)^\mu F_V\left[(\mathbf{p}_f - \mathbf{p}_i)^2\right] \,,$$

- calculated already in [Bijnens, Carloni '10]
- problem in cross-check with massless case in [Polyakov et al]
- independent study using disp.relations ($\pi\pi$ partial amplitude) [Bijnens,KK,Lanz '12]

$$\text{disc} F_V(s) = t_1^1 F_V(s)$$

- $\Rightarrow$ [Polyakov et al] wrong (typo...)

$O(N)$ sigma model: vector formfactor

[Bijnens,Carloni '10],[Bijnens,KK,Lanz '12]

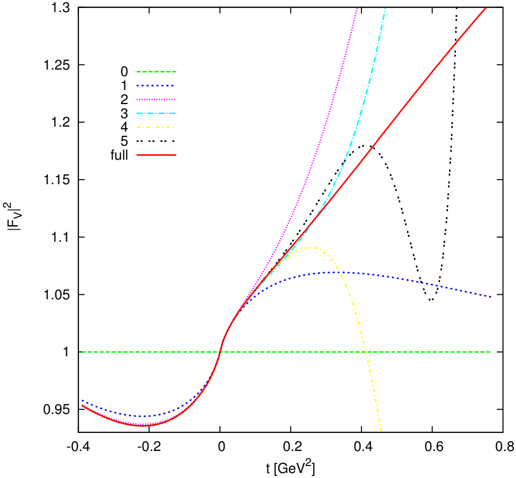
- calculated up to 5 loops
- chiral limit (see [Polyakov et al])
- closed form for NLL

$$F_V^{0NLL}(t) = 1 + \frac{1}{N} + \frac{4}{K_t N^2}\left[1 - \left(1 + \frac{2}{K_t N}\right)\log\left(1 + \frac{K_t N}{2}\right)\right]$$

$$K_t \equiv \frac{t}{16\pi^2 F^2}\log(-\mu^2/t)$$

$O(N)$ sigma model: vector formfactor

The expansions of the leading logarithms order by order for the vector form-factor in the chiral limit and next-to-large N limit ($F = 0.090$ GeV, $\mu = 0.77$ GeV and $N = 3$



Odd sector

unfortunately only for $N = 3$

1. $\pi\gamma \to \pi\pi$
2. $\pi \to \gamma\gamma$

Odd sector: 1. $\pi\gamma \to \pi\pi$

- $\pi^0\gamma \to \pi^0\pi^0$: C forbidden
- $\pi^-(p_1)\gamma(k) \to \pi^-(p_2)\pi^0(p_0)$ described by $VAAA$ box anomaly
- Mandelstam variables: s, t, u

$$s + t + u = 3M_\pi^2 + k^2$$

- due to symmetry, reasonable to take

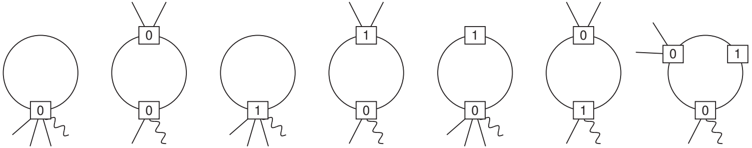
$$\Delta_n = s^n + t^n + u^n$$

-

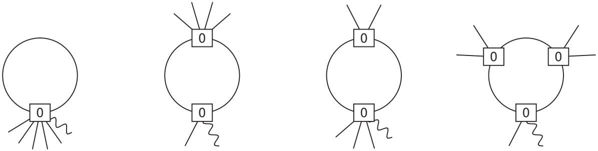
$$F^{3\pi} = F_0^{3\pi} f(s,t,u), \qquad F_0^{3\pi} = \frac{e}{4\pi^2 F_\pi^3}$$

Odd sector: 1. $\pi\gamma \to \pi\pi$

The irreducible diagrams for the process $\pi\gamma \to \pi\pi$ up to two-loop level



auxiliary diagrams



Anomln sektor: 1. $\pi\gamma \to \pi\pi$, vsledek

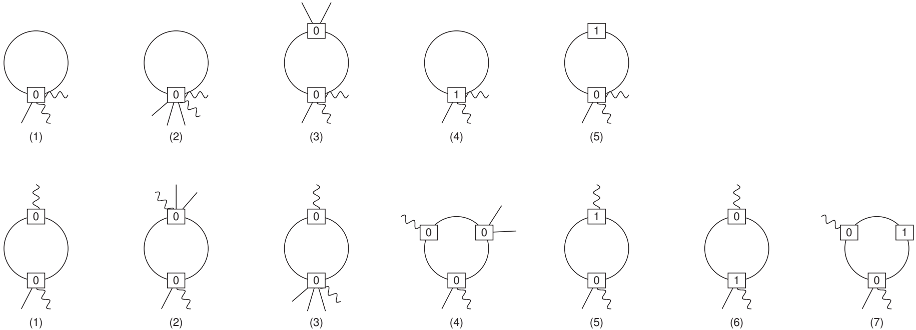
$$
\begin{aligned}
f^{LL}(s,t,u) = {} & 1 + L_{\mathcal{M}}\frac{1}{6}\left(3+\tilde{k}^2\right) + L_{\mathcal{M}}^2\frac{1}{72}(\tilde{k}^2-3)(\tilde{k}^2+33) \\
& + L_{\mathcal{M}}^3\frac{1}{1296}\left(90\tilde{\Delta}_3 - 640\tilde{\Delta}_2 - 8157 + 2105\tilde{k}^2 + 81\tilde{k}^4 + \tilde{k}^6\right) + L_{\mathcal{M}}^4\frac{1}{155520}\Big[-1532\tilde{\Delta}_4 \\
& + \tilde{\Delta}_3(88538 + 1890\tilde{k}^2) - \tilde{\Delta}_2(577760 + 12240\tilde{k}^2 + 540\tilde{k}^4) - 2433375 + 1296190\tilde{k}^2 \\
& + 57430\tilde{k}^4 + 480\tilde{k}^6 + 185\tilde{k}^8\Big] + L_{\mathcal{M}}^5\frac{1}{326592000}\Big[\tilde{\Delta}_5(13252156) \\
& - \tilde{\Delta}_4(160744570 + 518350\tilde{k}^2) + \tilde{\Delta}_3(1465187530 + 39593272\tilde{k}^2 + 247260\tilde{k}^4) \\
& - \tilde{\Delta}_2(6756522937 + 257781206\tilde{k}^2 + 11188776\tilde{k}^4 - 9160\tilde{k}^6) - 6498695163 \\
& + 12675091794\tilde{k}^2 + 801259373\tilde{k}^4 + 4780240\tilde{k}^6 + 2948600\tilde{k}^8 - 1832\tilde{k}^{10}\Big].
\end{aligned}
$$

remark: massless limit simple, no function found so-far

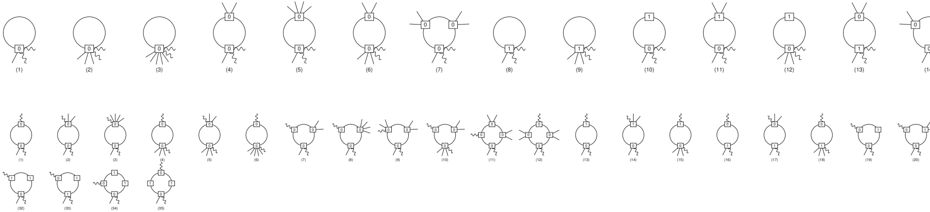
Odd sector: 2. $\pi^0\gamma\gamma$

- the most important process of this sector
- important in normalization of π^0 decays
- topology complicated: two types of one-loop diagrams
- no logarithms at one-loop diagrams [Donoghue et al'86], [Bijnens et al'88]
- true only for on-shell case
- logarithms at 2-loop order [KK,Moussallam'09]

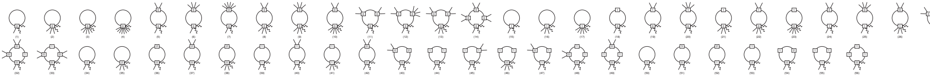
Odd sector: 2. $\pi^0\gamma\gamma$, 2 loop

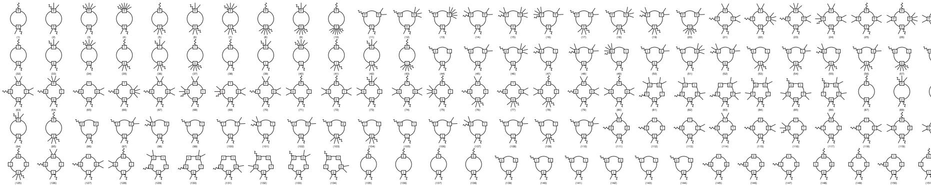


Odd sector: 2. $\pi^0\gamma\gamma$, 3 loop

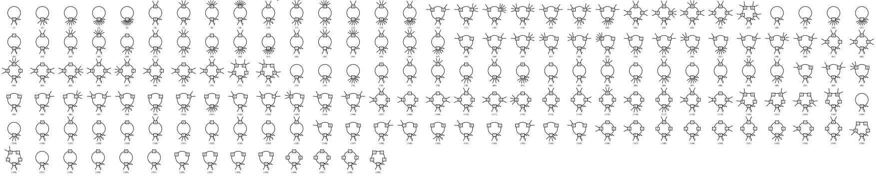


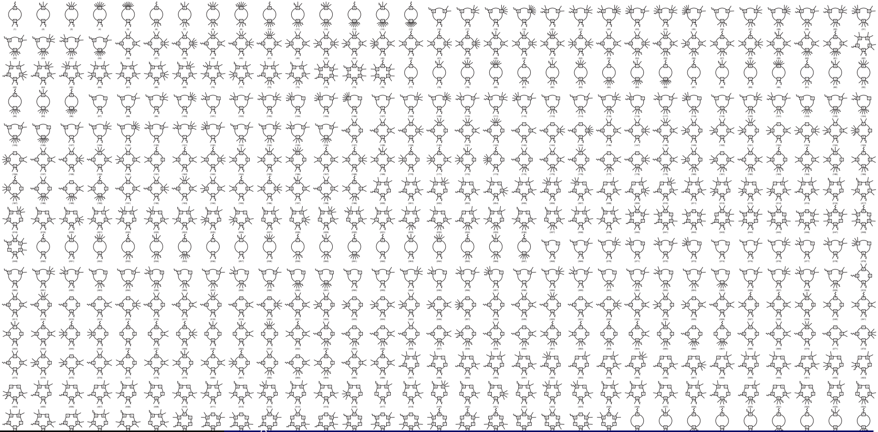
Odd sector: 2. $\pi^0\gamma\gamma$, 4 loop





Odd sector: 2. $\pi^0\gamma\gamma$, 5 loop





Odd sector: 2. $\pi^0\gamma\gamma$, result

$$F_{\pi\gamma\gamma}(k_1^2,k_2^2) = \frac{e^2}{4\pi^2 F_\pi} F_\gamma(k_1^2) F_\gamma(k_2^2) F_{\gamma\gamma}(k_1^2,k_2^2)\hat{F}\,.$$

$$\begin{aligned}
\hat{F} &= 1 - 1/6\,L_{\mathcal{M}}^2 + 5/6\,L_{\mathcal{M}}^3 + 56147/7776\,L_{\mathcal{M}}^4 + 446502199/11664000\,L_{\mathcal{M}}^5 \\
&\quad +65694012997/367416000\,L_{\mathcal{M}}^6\,, \\
F_\gamma(k^2) &= 1 + L_{\mathcal{M}}(1/6\,\tilde{k}^2) + L_{\mathcal{M}}^2(5/24\,\tilde{k}^2 + 1/72\,\tilde{k}^4) + L_{\mathcal{M}}^3(71/432\,\tilde{k}^2 + 1/24\,\tilde{k}^4 \\
&\quad +1/1296\,\tilde{k}^6) + L_{\mathcal{M}}^4(-24353/31104\,\tilde{k}^2 + 4873/10368\,\tilde{k}^4 - 2357/31104\,\tilde{k}^6 \\
&\quad +145/31104\,\tilde{k}^8) + L_{\mathcal{M}}^5(-548440741/81648000\,\tilde{k}^2 + 9793363/3024000\,\tilde{k}^4 \\
&\quad -32952389/54432000\,\tilde{k}^6 + 487493/13608000\,\tilde{k}^8 - 2069/10886400\,\tilde{k}^{10})\,, \\
F_{\gamma\gamma}(k_1^2,k_2^2) &= 1 + L_{\mathcal{M}}^3\tilde{k}_1^2\tilde{k}_2^2\frac{1}{72} + L_{\mathcal{M}}^4\tilde{k}_1^2\tilde{k}_2^2\big[-203/7776 + 29/10368(\tilde{k}_1^2+\tilde{k}_2^2) \\
&\quad +1/216(\tilde{k}_1^4+\tilde{k}_2^4) - 1/144\tilde{k}_1^2\tilde{k}_2^2\big] + L_{\mathcal{M}}^5\tilde{k}_1^2\tilde{k}_2^2\big[-5983633/10206000 \\
&\quad +46103/1632960(\tilde{k}_1^2+\tilde{k}_2^2) + 372113/11664000(\tilde{k}_1^4+\tilde{k}_2^4) \\
&\quad -211/5443200(\tilde{k}_1^6+\tilde{k}_2^6) - 394157/9072000\tilde{k}_1^2\tilde{k}_2^2 \\
&\quad -4/25515\tilde{k}_1^2\tilde{k}_2^2(\tilde{k}_1^2+\tilde{k}_2^2)\big]\,.
\end{aligned}$$

Phenomenology, 1. $\pi\gamma \to \pi\pi$

- little experimental information
- officially only experiment at Serpukhov

$$\pi^- + (Z, A) \to \pi^- + \pi^0 + (Z, A) \ . \tag{3}$$

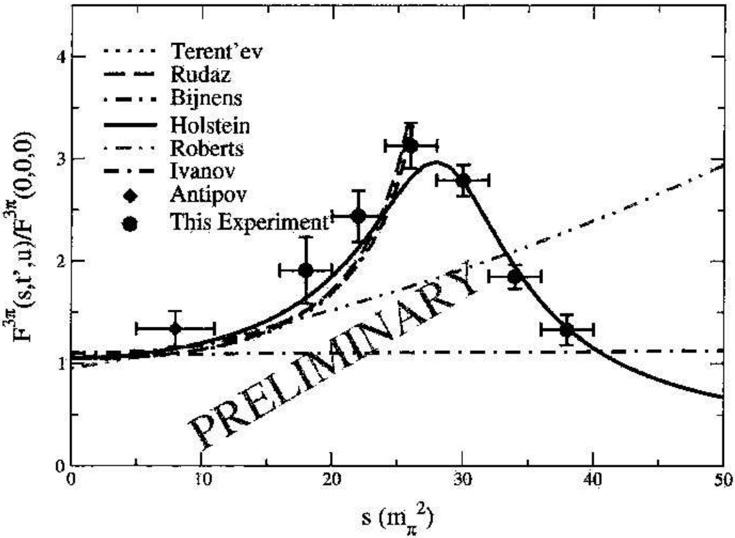
- Their analysis only includes events with low invariant mass of the final state pion pair ($\sqrt{s} < \sqrt{10} M_\pi$)
- the measurement indicates

$$\bar{F}^{3\pi}_{\text{exp}} = 12.9 \pm 0.9 \pm 0.5 \text{ GeV}^{-3}$$

- theoretical value $F_0^{3\pi} = 9.8$ GeV^{-3}
- new activity (?)
  - via $\pi^- e^- \to \pi^- e^- \pi^0$ (SPS CERN '85) [Giller,Scherer et al. '05]
  - CLAS in CEBAF at JLab

Phenomenology, 1. $\pi\gamma \to \pi\pi$

preliminary from CLAS [Miskimen, Asavapibhop '01]



Phenomenology, 1. $\pi\gamma \to \pi\pi$

- higher order corrections

$$F^{3\pi}(s,t,u) = F_0^{3\pi}(1 + f_0^{\mathsf{EM}} + f_1^{\mathsf{loop}} + f_1^{\mathsf{ct}} + \dots)\,.$$

- 1-loop corrections: [Bijnens, Bramon, Cornet '89]
- EM corrections: [Ametller, Knecht, Talavera '01]
- different models for counter-terms, e.g. hidden local symmetry (HLS), quark constituent model (QCM) [Strandberg '03], Schwinger-Dyson equation (SDE) [Jiang,Wang '10],
- The experimental extraction of the anomalous $\gamma 3\pi$ factor $F_0^{3\pi}$ (in GeV^{-3}) using various models for higher order corrections

	LO	f_0^{EM}	f_1^{loop}(LL)	f_1^{loop}	f_1^{ct} (HLS)	f_1^{ct} (QCM)	f_1^{ct} (SDE)
$F_0^{3\pi}$	12.9	12.3	12.0	11.8	11.3	10.1	11.9

Phenomenology, 1. $\pi\gamma \to \pi\pi$, LL

- The contributions from the leading logarithms up to sixth order to the experimental extraction of the anomalous $\gamma 3\pi$ factor $F_0^{3\pi}$ (in GeV^{-3}).

	LO	Δ1-LL	Δ2-LL	Δ3-LL	Δ4-LL	Δ5-LL
$F_0^{3\pi}$	12.9	-0.3	0.04	0.02	0.006	0.001

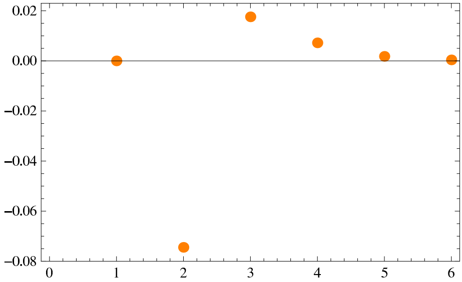
- energy dependence via some expansion, e.g. in s around $\pi\pi$ threshold

$$F^{3\pi}(s, t = 4M_\pi^2) = F_0^{3\pi}(1 + \alpha + \beta s/M_\pi^2 + O(s^2))$$

	α	β
c_1	$1/2$	0
c_2	$-11/8$	0
c_3	$-13367/1296$	$-775/648$
c_4	$-1414225/31104$	$-237877/25920$
c_5	$-14201792401/81648000$	$-4652736041/81648000$

Phenomenology, 2. $\pi^0 \to \gamma\gamma$

Leading logarithm contribution of individual orders in percent of the leading order:



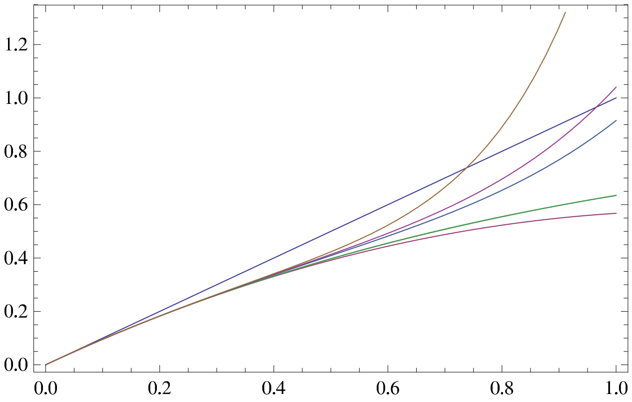
Adler-Lee-Treiman-Zee-Terentev theorem on triangle and box anomaly

$$F^{3\pi}(0,0,0) = \frac{1}{eF_\pi^2} F_{\pi\gamma\gamma}(0,0)$$

is valid up to 2-loop order for LL beyond the soft-photon limit

Phenomenology, 2. $\pi^0 \to \gamma^* \gamma$

Contribution of the LLs to $Q^2 F_\gamma(-Q^2)$ at different orders.



Phenomenology, 2. $\pi^0 \to \gamma^*\gamma$

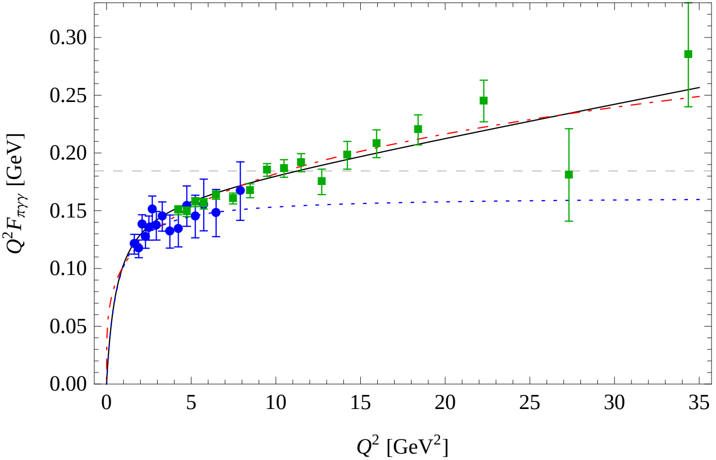


Figure: CLEO (blue points) and BABAR (green squares) data with fitted function

MesonNet