A Higher Order Duality Relation between Loops and Trees

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in collaboration with

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Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 09(2008)064

IB, Catani, Draggiotis, Rodrigo, JHEP 10(2010)073

Overview

- ⇝ **Why diagrams with many loops and many legs**
- ⇝ **Some Cutting–methods**
- ⇝ **Feynman's tree theorem**
- ⇝ **By–passing Feynman's Tree Theorem: The** *duality relation* **at one loop**
- ⇝ **The duality relation to higher loop order**
- ⇝ **Conclusion and Outlook**

Why diagrams with many loops and legs

The LHC is a hadron collider which is working at higher energies than ever reached before

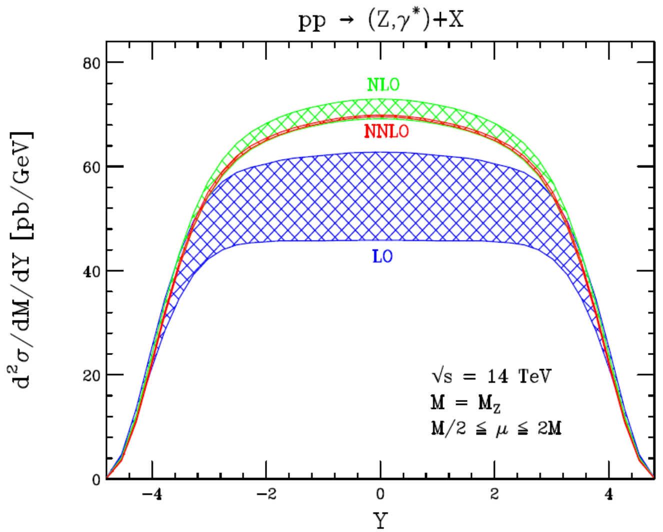
→ higher multiplicities (# legs): more powers of α_s

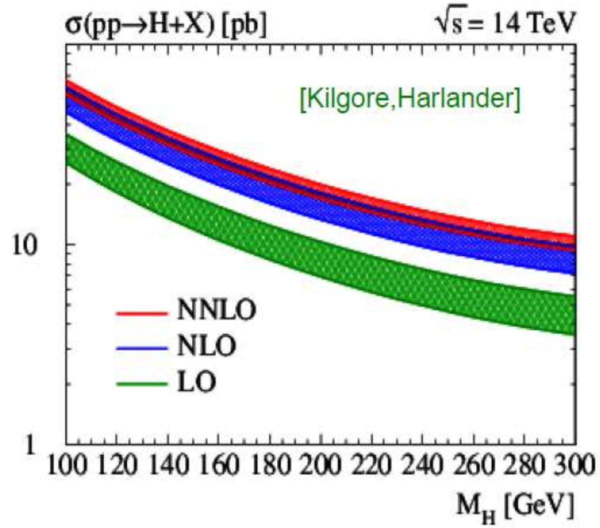
→ proton is not elementary: new channels might open at NLO

→ huge soft and collinear corrections & logs of the ratios of different scales

Higher orders systematically improve the precision of the theoretical predicitions (estimated by varying the renormalization/factorization scales) for background and signal processes

[Anastasiou, Dixon, Melnikov, Petriello]





Huge radiative corrections (QCD)

$$\int d\sigma^{NLO} = \int_m d\sigma^V + \int_{m+1} d\sigma^R$$

Combines integration over phase–space with different number of partons;
Kinoshita–Lee–Nauenberg: cancellation of IR poles

⟶ Real radiation

$$\int_{m+1} d\sigma^R = \int d\Phi^{(m+1)}(\{p_i\}) \times M^{(m+1)}(\{p_i\}) \times F^{(m+1)}(\{p_i\})$$

Split phase-space integrand in two parts: $(...)_{divergent} + (...)_{finite}$

IR singular: analytically up to $O(\varepsilon^{-1})$; **IR finite:** numerically as LO

Several well known/tested working methods (subtraction, dipole, slicing, mixed,...)

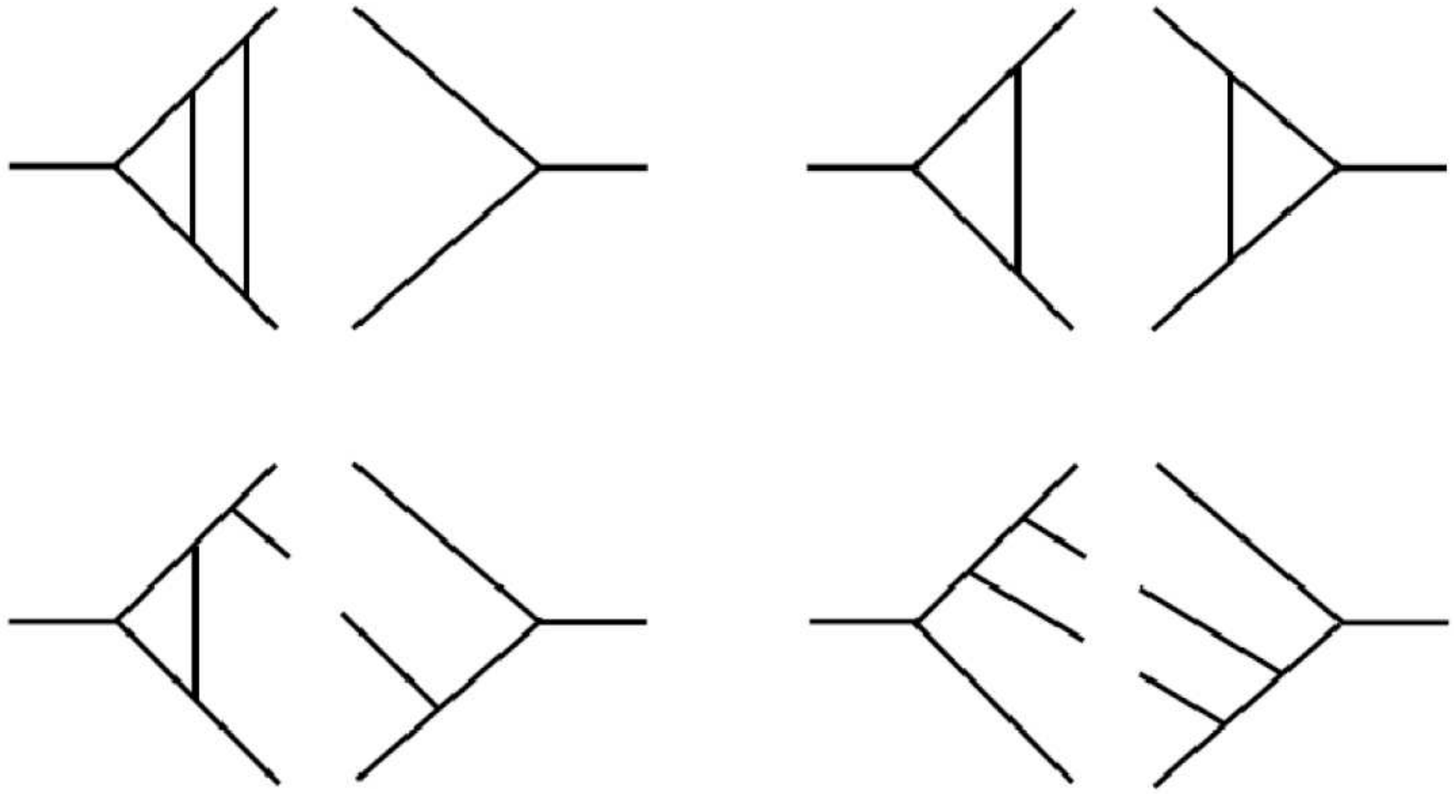
⟶ Virtual contribution

$$\int_m d\sigma^V = \int d\Phi^{(m)}(\{p_i\}) \times \int d^d q\, M^{(m)}(\{p_i\}) \times F^{(m)}(\{p_i\})$$

Loop integral: in multiparton processes ($m \geq 5$) regarded as main bottleneck!

NNLO

$$\int d\sigma^{NNLO} = \int_m d\sigma^V + \int_{m+1} d\sigma^{VR} + \int_{m+2} d\sigma^R$$



How to calculate the loops?

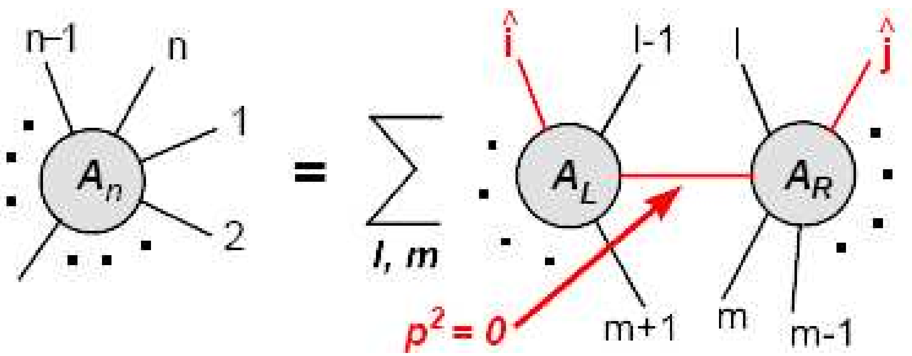
- Feynman diagrams: still a good way to go, but
  - Number of Feynman diagrams increases factorially with the number of external fields (legs)
  - Passarino–Veltmann reduction to scalar integrals:
    proliferation of spurious divergences (Gram determinants)

- Many new developments in recent years:
  Recursion relations and (generalized) unitarity
  Properties of the S–Matrix:
  Analyticity: scattering amplitudes are determined by their singularities
  Unitarity: the residues at singular points are products of scattering amplitudes with lower number of legs and/or less loops

BCFW = [Britto, Cachazo, Feng, Witten]

On–shell recursion relations at tree level. Reconstruct the scattering amplitude from its singularities: Add $z\eta^\mu$ (z complex) to the four-momentum of one external particle and subtract it on another such that the shift leaves them on-shell.

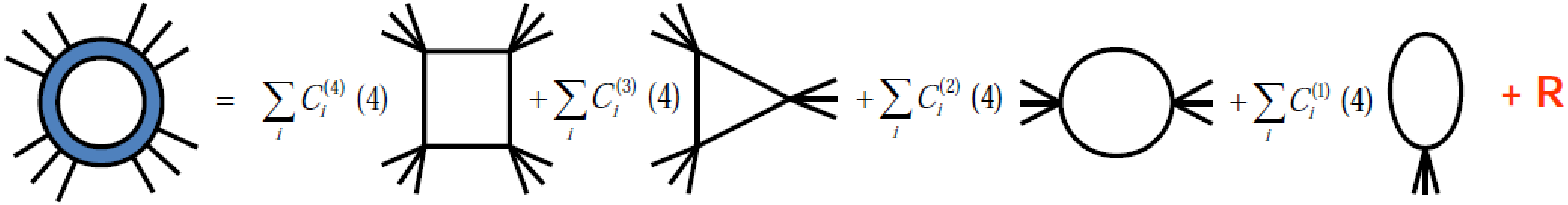
$$0 = \frac{1}{2\pi}\oint_{C\to\infty}\frac{A(z)}{z} = A(0) - \sum_{z_i}\frac{\mathrm{Res}_{z_i}(A(z))}{z_i}$$



Diagrammatic proof for gluon amplitudes [Draggiotis, Kleiss, Lazopoulos, Papadopoulos]

Generalized unitarity

A dimensionally regularized n–point one–loop integral (scattering amplitude) is a linear combination of boxes, triangles, bubbles and tadpoles with rational coefficients

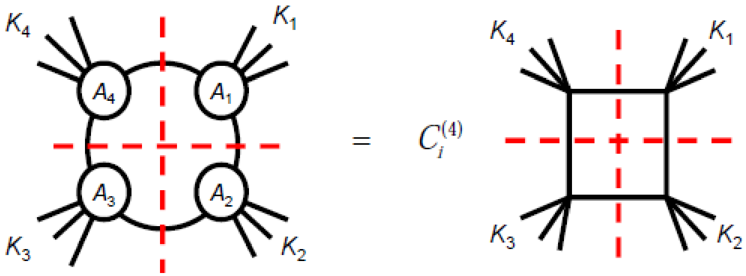


Pentagons and higher n–point functions can be reduced to lower point integrals and higher dimensional polygons that only contribute at $O(\varepsilon)$ [Bern, Dixon, Kosower]

⇒ The task is reduced to determining the coefficients: by applying multiple cuts at both sides of the equation [Britto, Cachazo, Feng]

R is a finite piece that is entirely rational: can not be detected by four–dimensional cuts.

Quadruple cut:

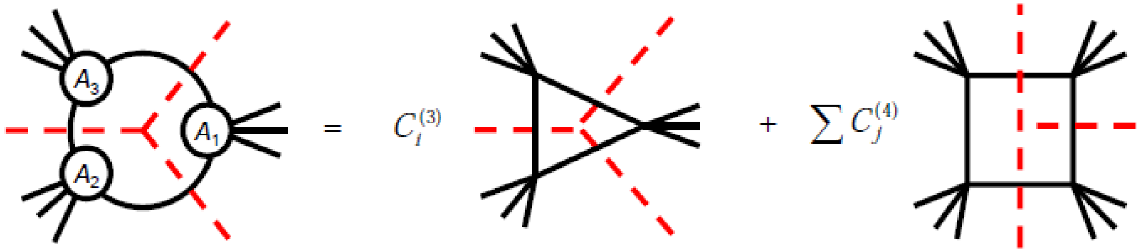


The discontinuity across the leading singularity is unique

$$C_i^{(4)} = A_1 \times A_2 \times A_3 \times A_4$$

Four on–shell constraints
$\Rightarrow$ freeze the loop momenta

Triple cut:



Only three on–shell constrains
$\Rightarrow$ one free component of the loop momentum

And so on for double and single cuts.

$\longrightarrow$ **OPP** [Ossola, Pittau, Papadopoulos]: a systematic way to extract the coefficients

Rational terms: d–dimensional cuts, recursion relations (BCFW), Feynman rules ...

Progress in the past years

Some examples for calculated processes in the recent years:
Contribution by various people and projects...

Many groups worked on calculating the necessary $2 \to 3$ processes for the LHC:

[Dittmaier, Uwer, Weinzierl], [Dittmaier, Kallweit, Uwer], [Reina, Dawson, Jackson, Wackeroth], [Beenakker, Dittmaier, Krämer, Plümper, Spira, Zerwas], [Bern, Dixon, Kosower], [Binoth, Ossola, Papadopoulos, Pittau], [Lazopoulos, McElmurry, Melnikov, Petriello], ...

Now the focus starts turning towards $2 \to 4$:

$pp \to t\bar{t}b\bar{b}$	[Bredenstein, Denner, Dittmaier, Pozzorini]
$q\bar{q} \to b\bar{b}b\bar{b}$	[GOLEM: Binoth, Greiner, Guffanti, Reuter, Guillet, Reiter]
$gg \to t\bar{t} + 2g$	[Diakonidis, Tausk]

$gg \to gggg$	[Many groups by now...]
$pp \to t\bar{t}b\bar{b}$	[CutTools/OPP,Helac: Bevilacqua, Czakon, Papdopoulos, Pittau, Worek]
$pp \to t\bar{t} + 2jets$	[Helac: Bevilacqua et al.]
$pp \to W/Z + 3jets$	[Rocket: Ellis, Giele, Kunzst, Melnikov, Zanderighi], [BlackHat+Sherpa: Berger, Bern, Dixon, Febres Codero, Forde, Gleisberg, Ita, Kosower, Maitre]
$pp \to W^+W^+jj$	[Melia, Melnikov, Röntsch, Zanderighi]

and beyond: $pp \to W + 4jets$ [BlackHat+Sherpa]

...and even more is going on - apologies to the ones not stated here...

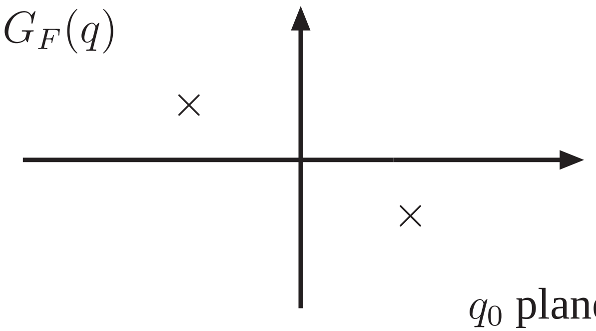
On–going attempts to higher orders

There are on–going attempts to find other methods and/or extend to higher loop orders.

Incomplete, completely biased example list of recent theoretical investigations for (higher–order) loop calculations based on cuts:

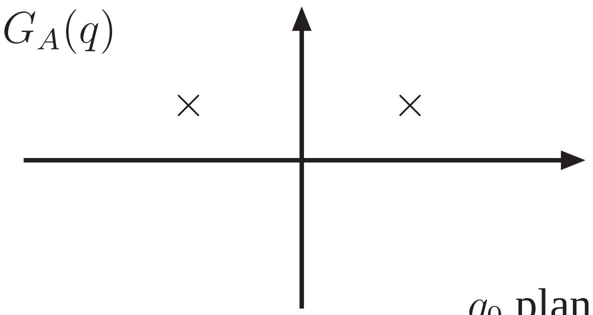
- ↠ Kilian, Kleinschmidt [arXiv:0912.3495]
- ↠ Caron–Huot [arXiv:1007.3224]
- ↠ Boels [arXiv:1008.3101]
- ↠ us: IB, Catani, Draggiotis, Rodrigo JHEP 10(2010)073, [arXiv:1007.0194]

The Feynman Tree Theorem



$$G_F(q) \equiv \frac{1}{q^2 + i0}$$

$+i0$: positive frequencies are propagated forward in time, negatives backward



$$G_A(q) \equiv \frac{1}{q^2 - i0\, q_0}$$

Both poles are placed above the real axis, independently of the sign of the energy

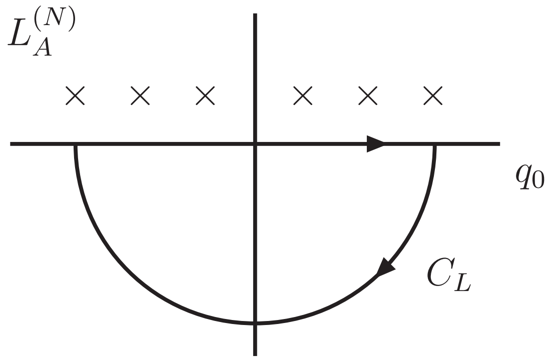
Both propagators are related by a delta function: $\frac{1}{x \pm i0} = \mathrm{PV}\left(\frac{1}{x}\right) \mp i\pi\,\delta(x)$

$$G_A(q) \equiv G_F(q) + \tilde{\delta}\,(q) \;\; , \quad \tilde{\delta}\,(q) \equiv 2\pi\, i\, \theta(q_0)\, \delta(q^2) = 2\pi\, i\; \delta_+(q^2)$$

$$G_R(q) \equiv G_F(q) + \tilde{\delta}\,(-q) \;\; , \quad G_F(-q) \equiv G_F(q)$$

The Feynman Tree Theorem

Advanced one–loop integral:
The integral along the given contour over advanced propagators vanishes



$$0 = L_A^{(1)}(p_1, p_2, \dots, p_N) = \int_q \prod_{i=1}^{N} G_A(q_i) = \int_q \prod_{i=1}^{N} \left[G_F(q_i) + \tilde{\delta}(q_i) \right]$$

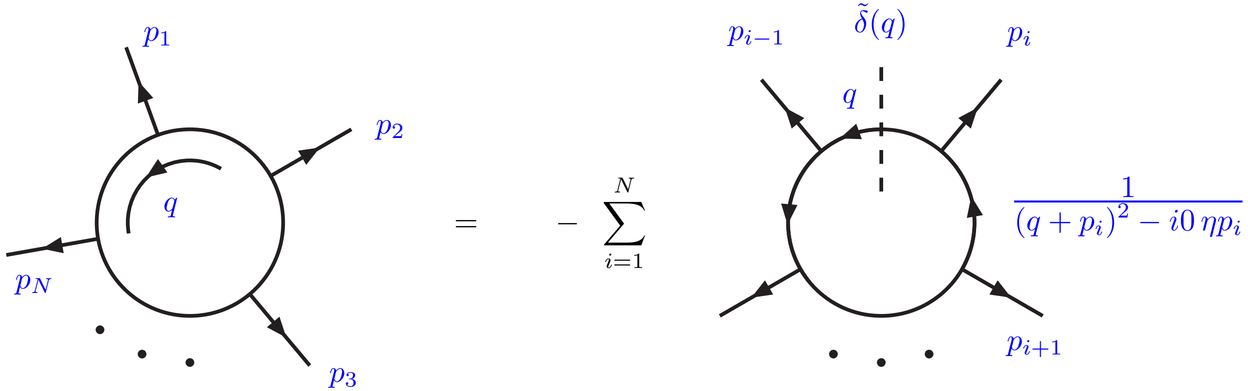
$$= L^{(1)}(p_1, p_2, \dots, p_N) + L^{(1)}_{1-\mathrm{cut}}(p_1, p_2, \dots, p_N) + \dots + L^{(1)}_{\mathrm{N-cut}}(p_1, p_2, \dots, p_N)$$

m–cut: $$L^{(1)}_{\mathrm{m-cut}}(p_1, p_2, \dots, p_N) = \int_q \left\{ \tilde{\delta}(q_1) \dots \tilde{\delta}(q_m)\; G_F(q_{m+1}) \dots G_F(q_N) + \text{uneq. perms.} \right\}$$

FTT: $$\boxed{L^{(1)}(p_1, p_2, \dots, p_N) = -\left[L^{(1)}_{1-\mathrm{cut}}(p_1, p_2, \dots, p_N) + \dots + L^{(1)}_{\mathrm{N-cut}}(p_1, p_2, \dots, p_N) \right]}$$

The one–loop integral is the sum of multiple–cut contributions (in $D = 4$: 4–cut maximum)

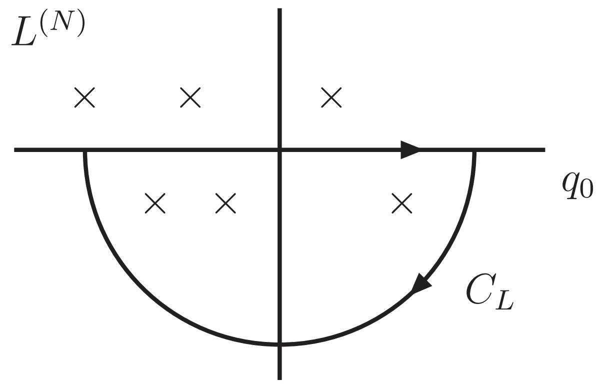
The duality relation [Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 09(2008)064], provides a single–cut relation for this expression by relating one–loop integrals (one–loop scattering amplitudes) with an arbitrary number of external legs (momenta) and corresponding *single–cut* Bremsstrahlung–Integrals:



- The duality relation recasts virtual corrections in a form that closely parallels the contribution of real corrections
- it is realised by modifying the customary "+i0" prescription of the Feynman propagators
- the new "+i0" prescription compensates for the absence of multiple–cut contributions that appear in the Feynman Tree Theorem

The duality relation at one–loop

Cauchy residue theorem in the loop energy complex plane selects residues with positive definite energy



$$L^{(1)}(p_1, p_2, \dots, p_N) = -2\pi i \int_{\mathbf{q}} \sum \mathrm{Res}_{\mathrm{Im}\, q_0 < 0} \left[\prod_{j=1}^{N} G_F(q_j) \right]$$

$$\mathrm{Res}_{\{i\text{–th pole}\}} \left[\prod_{j=1}^{N} G_F(q_j) \right] = \left[\mathrm{Res}_{\{i\text{–th pole}\}} \, G_F(q_i) \right] \left[\prod_{j \neq i} G_F(q_j) \right]_{\{i\text{–th pole}\}}$$

The duality relation at one–loop

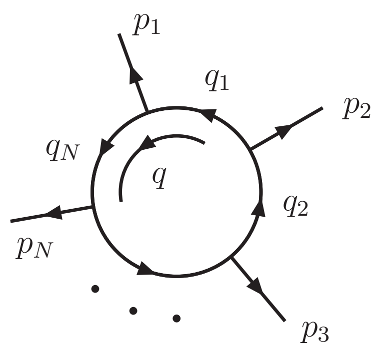
$$\left[\mathrm{Res}_{\{i\text{–th pole}\}} \frac{1}{q_i^2 + i0}\right] = \int dq_0 \; \delta_+(q_i^2)$$

- equivalent to cut that line and set it on–shell
- the one–loop integral can be represented as a linear combination of N single-cut phase–space integrals

$$\left[\prod_{j\neq i} G_F(q_j)\right]_{\{i\text{–th pole}\}} = \left[\prod_{j\neq i} \frac{1}{q_j^2 + i0}\right]_{\{q_i^2 = -i0\}} = \prod_{j\neq i} \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$

- equivalent to the shift $q_i^\mu \to q_i^\mu - i0 \frac{\eta^\mu}{2\eta q_i}$
- the customary $+i0$ prescription is modified
- Lorentz–covariant dual prescription
- η is a *future-like* vector: $\eta_\mu = (\eta_0, \boldsymbol{\eta})$, $\quad \eta_0 \geq 0, \; \eta^2 = \eta_\mu \eta^\mu \geq 0$
- different choices of η are equivalent to different choices of the coordinate system
- ONE integration momentum: $i0\,\eta(q_j - q_i)$ depends on external momenta only

The Duality Theorem at one loop:



$$L^{(1)}(p_1, ..., p_N) = -\tilde{L}^{(1)}_{1-cut}(p_1, ..., p_N) = -\sum_{i=1}^{N} \int_q \widetilde{\delta}\,(q_i) \prod_{j \neq i} G_D(q_i; q_j)$$

$$\widetilde{\delta}\,(q) \equiv 2\pi\, i\, \theta(q_0)\; \delta(q^2) = 2\pi\, i\; \delta_+(q^2)$$

The dual propagator: $G_D(q_i; q_j) := \dfrac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$

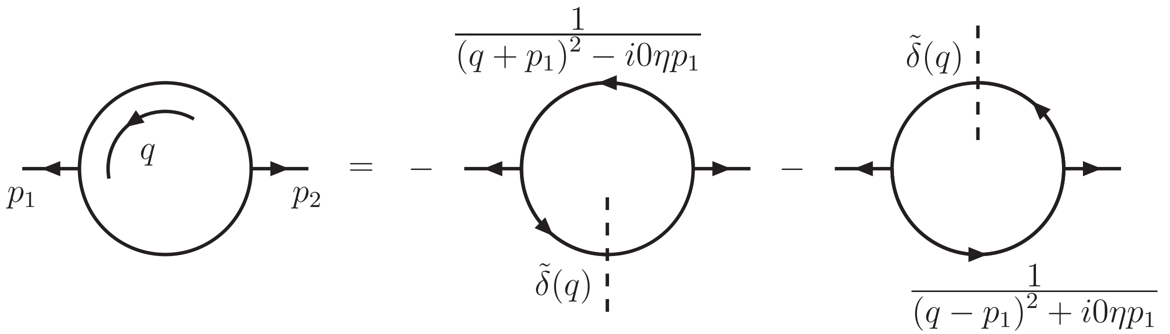
Individual cut integrals depend on η, *but the* η*–dependence cancels in the sum*

Duality and FTT

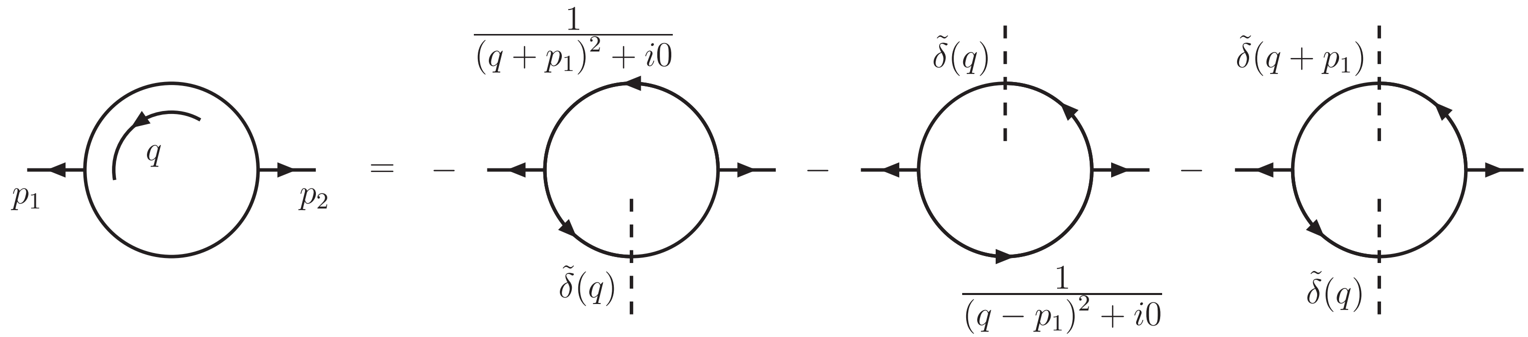
The duality and the FTT are equivalent: Dual and Feynman propagators are related by:

$$\tilde{\delta}(q_i)\; G_D(q_i;q_j) = \tilde{\delta}(q_i)\; \left[G_F(q_j) + \tilde{\theta}(q_j - q_i)\; \tilde{\delta}(q_j)\right], \qquad \tilde{\theta}(q) = \theta(\eta q)$$

Duality:



FTT:



Note: The double–cut contribution from the FTT is different from the unitarity cut that gives the imaginary part due to the different positive–energy flow of the internal lines.

Real masses (unitary theories) do not affect the dual prescription $\tilde{\delta}(q_i) \to \tilde{\delta}(q_i; M_i)$:

$$G_D(q_i; q_j) := \frac{1}{q_j^2 - M_j^2 - i0\,\eta(q_j - q_i)}$$

Unstable particles:

Dyson summation of self–energy → finite–width effect → *finite* imaginary contribution.

In the complex mass scheme:

$$G_C(q; s) := \frac{1}{q^2 - s} \quad , \quad s = \mathrm{Re}s + i\,\mathrm{Im}s, \quad \text{with} \quad \mathrm{Re}s > 0 > \mathrm{Im}s$$

produces poles in the q_0 plane that are located far from the real axis.

$$\tilde{L}^{(1)}(p_1, ..., p_N) \quad \to \quad \tilde{L}^{(1)}(p_1, ..., p_N) \quad + \quad \underbrace{\tilde{L}_C^{(1)}(p_1, ..., p_N)}_{\text{From the poles of the complex mass propagators}}$$

Complex mass $s(q^2)$, but always at a finite imaginary distance from the real axis.

The Effect of Gauges

Fictitious particles:

Fadeev–Popov ghosts in unbroken non–Abelian gauge theories, or would–be Goldstone bosons in spontaneously broken gauge theories ⇒ cut exactly as physical particles

Gauge boson: polarization tensor — t'Hooft-Feynman gauge $\xi = 1$ ✓

$$d^{\mu\nu} = -g^{\mu\nu} + (\xi - 1)\, \ell^{\mu\nu}(q)\, G_G(q)$$

$\ell^{\mu\nu}(q)$: harmless polynomial dependence on q

- Spontaneously–broken gauge theories

$$G_G(q) = \frac{1}{\xi(q^2 + i0) - M^2}$$

unitary gauge ($\xi = 0$) ✓

- Un–broken gauge theories
  - covariant gauge

$$G_G(q) = \frac{1}{(q^2 + i0)}$$

second order pole ↯

  - physical/axial gauge

$$G_G(q) = \frac{1}{(n \cdot q)^k}\,, \quad \mathrm{k} = 1, 2$$

if $n \cdot \eta = 0$ ✓

Since the duality theorem acts on the propagators only, it can be *extended from scalar integrals to full Feynman diagrams*, leaving all the other factors unchanged.

Interaction vertices introduce numerator factors, which are in local theories at most polynomials in the loop momentum
⇒ no additional singularities (with a convenient gauge choice),
unitarity constrains the confergence of the q_0–integration at infinity

For relativistic, local and unitary quantum field theories

$$\begin{aligned} \mathcal{A}^{(1-\text{loop})} &= -\tilde{\mathcal{A}}^{(1-\text{loop})} \\ &= -[\mathcal{A}^{(1-\text{loop})}_{1-\text{cut}} + \mathcal{A}^{(1-\text{loop})}_{2-\text{cut}} + \ldots + \mathcal{A}^{(1-\text{loop})}_{\text{N}-\text{cut}}] \end{aligned}$$

⇒ *Starting from one–loop scattering amplitude consider all the possible cuts and replace uncut propagators by dual propagators accordingly*

From tree–level amplitudes for the forward scattering process $P(q) \to P(q)$ in the field of N external legs ($N+2$ scattering amplitudes)

Duality at Higher Loop Order

To summarize the one–loop result:

- there exists only *one* integration momentum
- duality relation with only single cuts
- the $i0$–prescription of the dual propagator depends on external momenta only
  $\rightarrow$ no branch cuts

Can we obtain a similar duality relation at higher loop order, where there is more than one integration momentum, and in particular when the loops are overlapping?

$\longrightarrow$ *[IB, Catani, Draggiotis, Rodrigo, JHEP 10(2010)073]*

We still want:

- # cuts = # loops
- integration–momentum–independent $i0$–prescription

We will see that we have to relax one of these requirements!

Duality at Higher Loop Order

We first need to find a more abstract language:

In analogy to single propagators, define *for any set of (internal) momenta* $\alpha_k = \{q_0, q_1, ..., q_N\}$:

$$G_{F(A,R)}(\alpha_k) = \prod_{i\in\alpha_k} G_{F(A,R)}(q_i) \ , \qquad G_D(\alpha_k) = \sum_{i\in\alpha_k} \widetilde{\delta}(q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(q_i; q_j) \ .$$

with $G_D(\alpha_k) = \widetilde{\delta}(q_i)$ for $\alpha_k = \{q_i\}$.

For example, for $\alpha_k = \{q_1, q_2, q_3\}$:

$$G_D(\alpha_k) = \left[\, \widetilde{\delta}(q_1)\ G_D(q_1;q_2)\, G_D(q_1;q_3) + \widetilde{\delta}(q_2)\ G_D(q_2;q_1)\, G_D(q_2;q_3) + \widetilde{\delta}(q_3)\ G_D(q_3;q_1)\, G_D(q_3;q_2) \,\right]$$

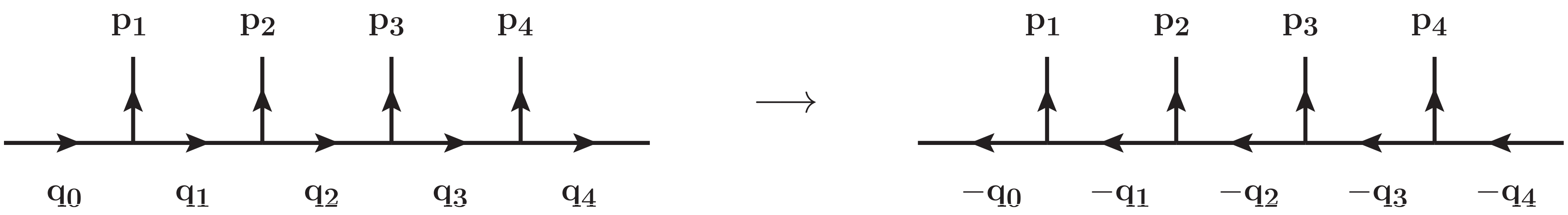
→ If the momenta in the set α_k depend on different integration momenta: integration–momentum dependence in $i0$–prescription

→ If the momenta in the set α_k depend on the same integration momentum: $i0$–prescription depends on external momenta only

⇒ *We will try to group the diagrams in the following in terms of inner lines, which depend on the same (combination of) integration momenta*

Duality at Higher Loop Order

We will also need $G_{F,A,R,D}(-\alpha_k)$ which means that the direction of momentum flow is reversed, $q_i \to -q_i$, for all momenta $q_i \in \alpha_k$:



$$G_F(-\alpha_k) = \prod_{i\in\alpha_k} G_F(-q_i) = G_F(\alpha_k)$$
$$G_A(-\alpha_k) = \prod_{i\in\alpha_k} G_A(-q_i) = G_R(\alpha_k)$$
$$G_D(-\alpha_k) = \sum_{i\in\alpha_k} \tilde{\delta}(-q_i) \prod_{\substack{j\in\alpha_k \\ j\neq i}} G_D(-q_i;-q_j) \; .$$

Note that $G_D(-\alpha_k)$ is not easily expressible in terms of $G_D(\alpha_k)$.

The relation between dual, advanced and Feynman propagators is given by:

$$\widetilde{\delta}(q_i)\ G_D(q_i, q_j) = \widetilde{\delta}(q_i)\ \left[G_F(q_j) + \tilde{\theta}(q_j - q_i)\ \widetilde{\delta}(q_j)\right], \qquad \tilde{\theta}(q) = \theta(\eta q)$$

$$G_A(q_i) = G_F(q_i) + \widetilde{\delta}(q_i) \quad .$$

From this we can obtain for **ANY** *set of (internal) momenta* α_k:

$$G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k) \qquad \textbf{Main Equation (I)}$$

This is a non–trivial relation relying on cancellations of theta–functions:

Set $\lambda_i = \eta(q_i - q_{i+1})$ for $i \in \{1, ..., n\}$, with $(n + i) \equiv i \bmod n$.
By construction, this fulfills (momentum conservation): $\sum_{i=1}^{n} \lambda_i = 0$, for which we find that

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \tilde{\theta}(q_i - q_j) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \tilde{\theta}(q_j - q_i) = 1 \ .$$

The "Multiplication Formula": How to express G_D in terms of subsets.

Consider $\beta_N \equiv \alpha_1 \cup ... \cup \alpha_N$:

$$\begin{aligned} G_D(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) &= G_A(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) - G_F(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) \\ &= \prod_{i=1}^{N} G_A(\alpha_i) - \prod_{i=1}^{N} G_F(\alpha_i) \\ &= \prod_{i=1}^{N} \left[G_F(\alpha_i) + G_D(\alpha_i) \right] - \prod_{i=1}^{N} G_F(\alpha_i) \end{aligned}$$

For example: $G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1)\, G_F(\alpha_2) + G_F(\alpha_1)\, G_D(\alpha_2) + G_D(\alpha_1)\, G_D(\alpha_2)$.

Main Equation (II)

$$G_D(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) = \sum_{\beta_N^{(1)} \cup \beta_N^{(2)} = \beta_N} \prod_{i_1 \in \beta_N^{(1)}} G_D(\alpha_{i_1}) \prod_{i_2 \in \beta_N^{(2)}} G_F(\alpha_{i_2}) \, .$$

The sum runs over all partitions of β_N into exactly two blocks $\beta_N^{(1)}$ and $\beta_N^{(2)}$ with elements $\alpha_i, i \in \{1, ..., N\}$, where we include the case: $\beta_N^{(1)} \equiv \beta_N$, $\beta_N^{(2)} \equiv \emptyset$.

One–loop revisited...

Now we have everything to extend the Duality Relation to higher loop orders. Start by revisiting the one–loop case:

$$0 = \int_{\ell_1} G_A(\alpha_1) = \int_{\ell_1} \left[G_F(\alpha_1) + G_D(\alpha_1) \right] \ ,$$

where α_1 labels *all* internal one–loop momenta q_i.

$$\boxed{L^{(1)}(p_1, \ldots, p_N) = - \int_{\ell_1} G_D(\alpha_1) \ .}$$

In this way, we directly obtain the duality relation between one–loop integrals and single–cut phase–space integrals

→ the above equation can be interpreted as the **application of the duality theorem to the given set of momenta** α_1 .

By definition of G_D, it obviously agrees with the one–loop result:

$$L^{(1)}(p_1, ..., p_N) \quad = \quad - \sum_{i=1}^{N} \int_q \widetilde{\delta}\,(q_i) \prod_{\substack{j=1 \\ j \neq i}} G_D(q_i; q_j)$$

Duality and FTT at one–loop

$$L^{(1)}(p_1,\ldots,p_N) = -\int_{\ell_1} G_D(\alpha_1)\ .$$

Using the multiplication formula for the set α_1 where the elements are given by all single propagators q_i, $\alpha_1 = q_1 \cup \ldots \cup q_N$:

$$G_D(q_1 \cup \ldots \cup q_N) \quad = \quad \sum_{\alpha^{(1)}\cup\alpha^{(2)}=\alpha} \prod_{i_1\in\alpha^{(1)}} G_D(q_{i_1}) \prod_{i_2\in\alpha^{(2)}} G_F(q_{i_2})\,,$$

and $G_D(q_i) = \widetilde{\delta}\,(q_i) \equiv$ one cut, we reproduce the FTT at one–loop:

$$\begin{aligned} L^{(1)}(p_1,\ldots,p_N) &= -\sum_{\alpha_1^{(1)}\cup\alpha_1^{(2)}=\alpha_1} \int_{\ell_1} \prod_{i_1\in\alpha_1^{(1)}} \widetilde{\delta}\,(q_{i_1}) \prod_{i_2\in\alpha_1^{(2)}} G_F(q_{i_2}) \\ &= -\left[\, L^{(1)}_{1\,\widetilde{\delta}(\mathrm{q_i})}(p_1,p_2,\ldots,p_N) + \cdots + L^{(1)}_{\mathrm{N}\,\widetilde{\delta}(\mathrm{q_i})}(p_1,p_2,\ldots,p_N)\,\right] \end{aligned}$$

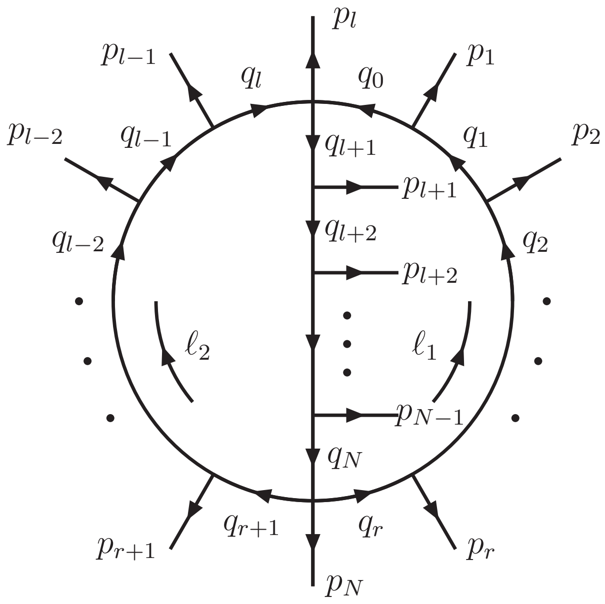
By definition the sum runs over all possible single– up to N–tuple cuts.
The m–cut integral of the FTT is given by the sum of the contributions from all partitions of α_1, with $\alpha_1^{(1)}$ containing precisely m elements.

How can we use all this to find a formula for higher order loops with the required properties?

The general idea which we would like to follow:

We apply the duality (sub)loop–by–(sub)loop to a higher order diagram using Equation I and express the result in terms of subsets which have the desired independence in their $i0$–prescription, using Equation II. ⇒ *This almost leads to the desired result*.

The correct subsets: Group lines with the same combination of integration momenta:



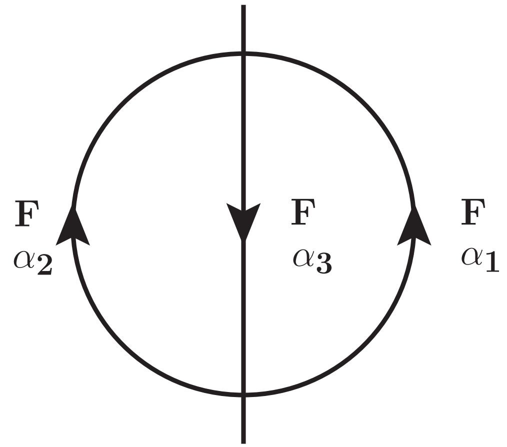
The "Loop Lines"

$$\alpha_1 \equiv \alpha_1(\ell_1) \equiv \{0, 1, ..., r\} \ ,$$
$$\alpha_2 \equiv \alpha_2(\ell_2) \equiv \{r+1, r+2, ..., l\} \ ,$$
$$\alpha_3 \equiv \alpha_3(\ell_1+\ell_2) \equiv \{l+1, l+2, ..., N\} \ .$$

Duality at Higher Loop Order

A two–loop example:

$$L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$
$$= -\int_{\ell_1} \int_{\ell_2} G_D(\alpha_1 \cup \alpha_3)\, G_F(\alpha_2)$$



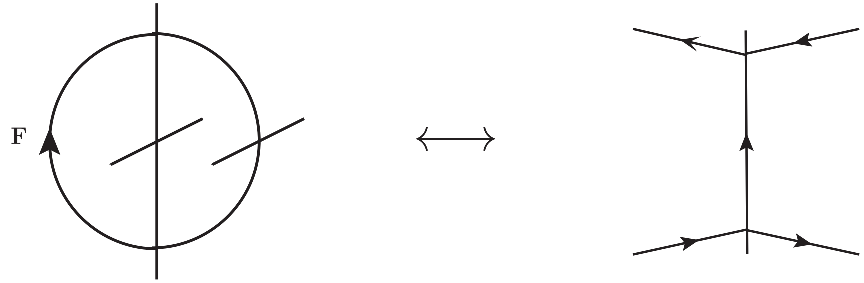
Use the multiplication formula for $G_D(\alpha_1 \cup \alpha_3)$:

$$L^{(2)}(p_1, p_2, \ldots, p_N)$$

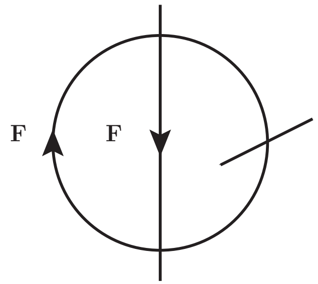
$$= -\int_{\ell_1} \int_{\ell_2} \left\{ G_D(\alpha_1)\, G_D(\alpha_3) + G_D(\alpha_1)\, G_F(\alpha_3) + G_F(\alpha_1)\, G_D(\alpha_3) \right\} G_F(\alpha_2)$$
$$= -\int_{\ell_1} \int_{\ell_2} \left\{ G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G_D(\alpha_1)\, G_F(\alpha_2 \cup \alpha_3) + G_F(\alpha_1 \cup \alpha_2)\, G_D(\alpha_3) \right\} .$$

The various cut contributions:

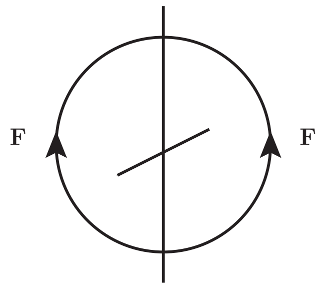
$G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3)$:



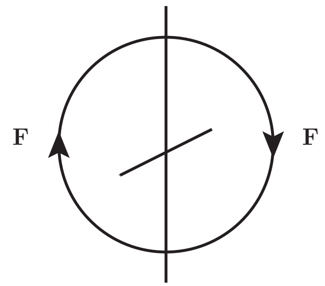
$G_D(\alpha_1)\, G_F(\alpha_2 \cup \alpha_3)$:



$G_F(\alpha_1 \cup \alpha_2)\, G_D(\alpha_3)$:



$\alpha_1 \to -\alpha_1$



$G_F(-\alpha_1 \cup \alpha_2)\, G_D(\alpha_3)$

Duality at Higher Loop Order

Two possible representations for the result:

$$L^{(2)}(p_1, p_2, \dots, p_N)$$

$$= \int_{\ell_1} \int_{\ell_2} \Big\{ - G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G_D(\alpha_1)\, G_D(\alpha_2 \cup \alpha_3) + G_D(\alpha_3)\, G_D(-\alpha_1 \cup \alpha_2) \Big\}$$

Formula with only double–cuts but integration momentum dependent $i0$–prescription.

$$L^{(2)}(p_1, p_2, \dots, p_N)$$
$$= \int_{\ell_1} \int_{\ell_2} \{ G_D(\alpha_1)\, G_D(\alpha_2)\, G_F(\alpha_3) + G_D(-\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3) + G^*(\alpha_1)\, G_D(\alpha_2)\, G_D(\alpha_3) \} \ ,$$

where $$G^*(\alpha_k) \equiv G_F(\alpha_k) + G_D(\alpha_k) + G_D(-\alpha_k) \ .$$

Formula with **triple cuts** *but integration–momentum–free $i0$–prescription.*

This can also be expressed as: $$G^*(\alpha_k) \equiv G_A(\alpha_k) + G_R(\alpha_k) - G_F(\alpha_k) \ .$$

Feynman Tree Theorem at Two Loops

Like in the one–loop case, we can also derive a FTT at two loops, using the multiplication formula:

$$L^{(2)}(p_1,\cdots,p_N)=\sum_{\substack{\alpha_k^{(1)}\cup\alpha_k^{(2)}=\alpha_k\\ k\in\{1,2,3\}}}\int_{\ell_1}\int_{\ell_2}\Bigg\{G_F(\alpha_1)\prod_{i_1\in\alpha_2^{(1)}}\tilde{\delta}(q_{i_1})\prod_{i_2\in\alpha_3^{(1)}}\tilde{\delta}(q_{i_2})\prod_{i_3\in\alpha_2^{(2)}\cup\alpha_3^{(2)}}G_F(q_{i_3})$$
$$+\,G_F(\alpha_2)\prod_{i_1\in\alpha_1^{(1)}}\tilde{\delta}(-q_{i_1})\prod_{i_2\in\alpha_3^{(1)}}\tilde{\delta}(q_{i_2})\prod_{i_3\in\alpha_1^{(2)}\cup\alpha_3^{(2)}}G_F(q_{i_3})$$
$$+\,G_F(\alpha_3)\prod_{i_1\in\alpha_1^{(1)}}\tilde{\delta}(q_{i_1})\prod_{i_2\in\alpha_2^{(1)}}\tilde{\delta}(q_{i_2})\prod_{i_3\in\alpha_1^{(2)}\cup\alpha_2^{(2)}}G_F(q_{i_3})$$
$$+\left(\prod_{i_1\in\alpha_1^{(1)}}\tilde{\delta}(q_{i_1})+\prod_{i_1\in\alpha_1^{(1)}}\tilde{\delta}(-q_{i_1})\right)\prod_{i_2\in\alpha_2^{(1)}}\tilde{\delta}(q_{i_2})\prod_{i_3\in\alpha_3^{(1)}}\tilde{\delta}(q_{i_3})\prod_{i_4\in\alpha_1^{(2)}\cup\alpha_2^{(2)}\cup\alpha_3^{(2)}}G_F(q_{i_4})\Bigg\}\,.$$

Duality at Higher Loop Order

⇒ The "algorithm" for higher orders (depending on what you want) :

(1) Identify the Loop Lines of a given Feynman Diagram

(2) Take a loop (= Loop Lines depending on the same integration momentum) of a higher order diagram and apply the Duality Relation to the corresponding Loop Lines $\alpha_1 \cup ... \cup \alpha_k$.

(3) Apply the multiplication formula, Equation II, to $G_D(\alpha_1 \cup ... \cup \alpha_k)$.

(4) Repeat steps (1) to (3) until the number of $G_D s$ in each term is equal to the number of Loops. If necessary, change the direction of momentum flow for certain Loop Lines.

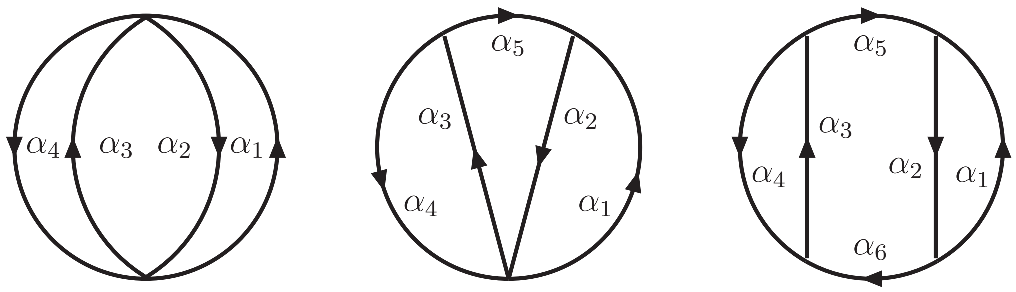
⇒ *One obtains a result in terms of tree diagrams, where the number of cuts is equal to the number of loops, but some of the propagators still have branch cuts.*

In case this is not what you want, continue:

(5) Identify the $G_D(\alpha_1 \cup ... \cup \alpha_k)$, which still contain sets of different Loop Lines and apply Equation II.

⇒ *One obtains a result in terms of disconnected tree diagrams, where all propagators have momentum–independent $i0$–prescription and the number of cuts ranges from the number of Loops to the number of Loop Lines.*

Duality at Three Loops



(a) (b) (c)

$$L^{(3)}_{(\mathrm{a}),(\mathrm{b}),(\mathrm{c})}(p_1, p_2, \ldots, p_N) = \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} G_D(\alpha_1 \cup \alpha_2)\, G_D(\alpha_3 \cup \alpha_4)\, G_F(\beta)$$

with (a): $\beta = \emptyset$, (b): $\beta = \alpha_5$, (c): $\beta = \alpha_5 \cup \alpha_6$

Use: $G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1)\, G_D(\alpha_2) + G_D(\alpha_1)\, G_F(\alpha_2) + G_F(\alpha_1)\, G_D(\alpha_2)$.

What is still missing is of the form:

$$\int_{\ell_1}\int_{\ell_2}\int_{\ell_3} G_D(\alpha_1) G_F(\alpha_2) G_F(\alpha_3) G_D(\alpha_4) \;\to\; -\int_{\ell_1}\int_{\ell_2}\int_{\ell_3} G_D(\alpha_1) G_D(\alpha_2 \cup \alpha_3) G_D(\alpha_4) \,.$$

One can obtain a triple–cut expression for these diagrams. This expression here consists of (mostly) triple cuts only, but contains momentum–dependent i0–prescription in the propagators of the last two lines!

$$
\begin{aligned}
L^{(3)}_{(a),(b),(c)}(p_1,p_2,\dots,p_N) &= \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} G_D(\alpha_1\cup\alpha_2)\, G_D(\alpha_3\cup\alpha_4)\, G_F(\beta) \\
= \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} \Big\{ \Big[& G_D(\alpha_2,\alpha_3,\alpha_4)\, G_F(\alpha_1) + G_D(\alpha_1,\alpha_3,\alpha_4)\, G_F(\alpha_2) + G_D(\alpha_1,\alpha_2,\alpha_4)\, G_F(\alpha_3) \\
& + G_D(\alpha_1,\alpha_2,\alpha_3)\, G_F(\alpha_4) + G_D(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \Big] G_F(\beta) \\
& - G_D(\alpha_1,\alpha_3)\, G_D(\alpha_2\cup -\alpha_4\cup\beta) - G_D(\alpha_1,\alpha_4)\, G_D(\alpha_2\cup\alpha_3\cup\beta) \\
& - G_D(\alpha_2,\alpha_3)\, G_D(-\alpha_1\cup -\alpha_4\cup\beta) - G_D(\alpha_2,\alpha_4)\, G_D(-\alpha_1\cup\alpha_3\cup\beta) \Big\}
\end{aligned}
$$

with $G_D(\alpha_1,...,\alpha_N) := \prod_{i=1}^{N} G_D(\alpha_i)$.

Duality at Three Loops

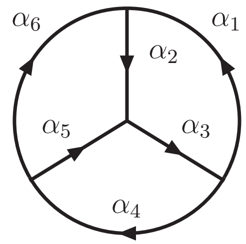
For the basket:

$$L^{(3)}_{\rm basket}(p_1,p_2,\dots,p_N) = -\int_{\ell_1}\int_{\ell_2}\int_{\ell_3} \Big\{ G_D(\alpha_2,\alpha_3,-\alpha_4)\,G_F(\alpha_1) + G_D(\alpha_1,\alpha_3,-\alpha_4)\,G_F(\alpha_2)$$
$$+ G_D(-\alpha_1,\alpha_2,\alpha_4)\,G_F(\alpha_3) + G_D(-\alpha_1,\alpha_2,\alpha_3)\,G_F(\alpha_4)$$
$$+ G_D(-\alpha_1,\alpha_2,\alpha_3,\alpha_4) + G_D(\alpha_1,\alpha_2,\alpha_3,-\alpha_4) + G_D(-\alpha_1,\alpha_2,\alpha_3,-\alpha_4) \Big\}\,.$$

This expression consists of triple (number of loops) and quadruple (number of loop lines) cuts, but momentum–independent $i0$–prescriptions in all propagators!

This can be found for any diagram considered so far:

- # cuts = # loops, but momentum–dependent $i0$–prescriptions
- # cuts range from # loops to # loop lines, momentum–independent $i0$–prescriptions



$$L^{(3)}_{\text{Mercedes}}(p_1, p_2, \ldots, p_N)$$

$$= \int_{\ell_1}\int_{\ell_2}\int_{\ell_3} \Big\{ - G_D(\alpha_1, \alpha_2, \alpha_3)\, G_F(\alpha_4, \alpha_5, \alpha_6) + G_D(\alpha_3 \cup \alpha_4 \cup \alpha_5)\, G_D(\alpha_1, \alpha_2)\, G_F(\alpha_6)$$

$$+ G_D(-\alpha_1 \cup \alpha_4 \cup \alpha_6)\, G_D(\alpha_2, \alpha_3)\, G_F(\alpha_5) + G_D(-\alpha_2 \cup \alpha_5 \cup -\alpha_6)\, G_D(\alpha_1, \alpha_3)\, G_F(\alpha_4)$$

$$+ G_D(\alpha_1)[G_D(\alpha_3 \cup \alpha_4)\, G_D(\alpha_5)\, G_F(\alpha_2 \cup \alpha_6) - G_D(\alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_6)\, G_D(\alpha_5)$$

$$- G_D(\alpha_3 \cup \alpha_4)\, G_D(-\alpha_2 \cup \alpha_5 \cup -\alpha_6)]$$

$$+ G_D(\alpha_2)[G_D(-\alpha_1 \cup \alpha_6)\, G_D(\alpha_4)\, G_F(\alpha_3 \cup \alpha_5) - G_D(\alpha_1 \cup \alpha_3 \cup \alpha_5 \cup -\alpha_6)\, G_D(\alpha_4)$$

$$- G_D(-\alpha_1 \cup \alpha_6)\, G_D(\alpha_3 \cup \alpha_4 \cup \alpha_5)]$$

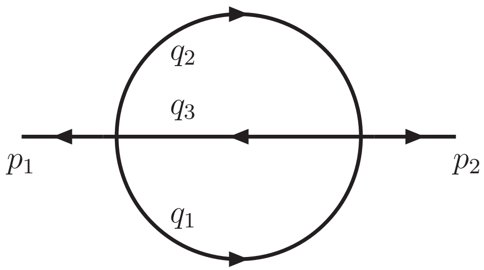
$$+ G_D(\alpha_3)[G_D(-\alpha_2 \cup \alpha_5)\, G_D(-\alpha_6)\, G_F(\alpha_1 \cup \alpha_4) - G_D(-\alpha_1 \cup -\alpha_2 \cup \alpha_4 \cup \alpha_5)\, G_D(-\alpha_6)$$

$$- G_D(-\alpha_2 \cup \alpha_5)\, G_D(-\alpha_1 \cup \alpha_4 \cup \alpha_6)] \Big\} .$$

Conclusion and Outlook

- Done:
  - We developed a relation between loop integrals and real radiation phase–space integrals to higher loop orders
  - The results can be expressed in two different ways in terms of cuts:
    * # cuts = # loops, $i0$–prescription integration–momentum dependent
    * # cuts = # loops to # Loop Lines, $i0$–prescription integration–momentum free
  - Everything is true for diagrams with single–pole propagators
- Next steps:
  - On the theoretical side:
    * Investigate the IR– and UV–structure of the expressions/integrals
    * How is all this explicitly looking on the scattering amplitude level
    * Higher order poles
  - Numerical implementation:
    * Calculate a full example process to investigate numerical stability and effectiveness

Calculation of a two–loop example: *the massless two–loop two–point sun–rise*



$$L^{(2)}(p_1,p_2) = \int_{\ell_1}\int_{\ell_2} \left\{ \widetilde{\delta}\,(q_1)\;\widetilde{\delta}\,(q_2)\;G_F(q_3) + \widetilde{\delta}\,(-q_1)\;G_F(q_2)\,\widetilde{\delta}\,(q_3) + G^*(q_1)\,\widetilde{\delta}\,(q_2)\;\widetilde{\delta}\,(q_3) \right\} \;.$$

With $q_1 = \ell_1$, $q_2 = \ell_2$ and $q_3 = \ell_1 + \ell_2 + p_1$.

Replacing $G^*(q_1) = G_F(q_1) + \widetilde{\delta}\,(q_1) + \widetilde{\delta}\,(-q_1)$, and shifting some momenta, we obtain

$$\begin{aligned} L^{(2)}(p_1,p_2) \quad &= \quad \int_{\ell_1}\int_{\ell_2} \widetilde{\delta}\,(\ell_1)\;\widetilde{\delta}\,(\ell_2)\;\Big\{ G_F(\ell_1+\ell_2+p_1) + G_F(\ell_1+\ell_2-p_1) + G_F(\ell_1-\ell_2-p_1) \\ &\qquad + \widetilde{\delta}\,(\ell_1+\ell_2+p_1) + \widetilde{\delta}\,(\ell_1+\ell_2-p_1) \Big\} \;. \end{aligned}$$

Back–up slide: Example

$$\int_{\ell_1} \widetilde{\delta}\,(\ell_1)\; G_F(\ell_1+k) \;=\; d_\Gamma\,\left[k^2+i0\right]^{-\epsilon}\left[1+\theta(k^2)\,\theta(-k_0)\,\left(e^{i2\pi\epsilon}-1\right)\right] \;,$$

$$\int_{\ell_1} \widetilde{\delta}\,(\ell_1)\; \widetilde{\delta}\,(\ell_1+k) \;=\; d_\Gamma\,\left[k^2+i0\right]^{-\epsilon}\theta(-k^2)\,\left(e^{i2\pi\epsilon}-1\right) \;,$$

$$d_\Gamma = -\frac{c_\Gamma}{2}\,\frac{1}{\epsilon(1-2\epsilon)}\,\frac{1}{\cos(\pi\epsilon)}\;, \qquad c_\Gamma = \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\,\Gamma(1-2\epsilon)}\;,$$

$$\begin{aligned} L^{(2)}(p_1,p_2) \;=\;& d_\Gamma \int_{\ell_2} \widetilde{\delta}\,(\ell_2)\,\Big\{\left[(\ell_2+p_1)^2+i0\right]^{-\epsilon}\left(e^{i2\pi\epsilon}+1\right) \\ &+\left[(\ell_2-p_1)^2+i0\right]^{-\epsilon}\left[e^{i2\pi\epsilon}-\theta((\ell_2-p_1)^2)\theta((\ell_2-p_1)_0)\left(e^{i2\pi\epsilon}-1\right)\right]\Big\}\;. \end{aligned}$$

$$d_\Gamma \int_{\ell_2} \widetilde{\delta}\,(\ell_2)\,\left[(\ell_2+k)^2+i0\right]^{-\epsilon} =$$
$$-\,G_2\,\frac{\sin(\pi\epsilon)\,e^{-i2\pi\epsilon}}{\sin(3\pi\epsilon)}\,(-k^2-i0)^{1-2\epsilon}\left[1+\theta(k^2)\theta(-k_0)\left(e^{i2\pi\epsilon}-1\right)\right]\;,$$

and

$$d_\Gamma \int_{\ell_2} \widetilde{\delta}\,(\ell_2)\,\left[(\ell_2+k)^2+i0\right]^{-\epsilon}\,\theta((\ell_2+k)^2)\,\theta((\ell_2+k)_0) =$$
$$G_2\,\frac{\sin(\pi\epsilon)}{\sin(3\pi\epsilon)}\,(-k^2-i0)^{1-2\epsilon}\left[\theta(-k^2)-\theta(k^2)\,\theta(k_0)\,e^{-i2\pi\epsilon}\right]\;,$$

where

$$G_2 = \frac{\Gamma(-1+2\epsilon)\,\Gamma(1-\epsilon)^3}{(4\pi)^{4-2\epsilon}\,\Gamma(3-3\epsilon)}\;.$$

$$L^{(2)}(p_1,p_2) = -\,G_2\,(-p_1^2-i0)^{1-2\epsilon}\;,$$

which is the well–known result for the massless sunrise two–loop two–point function.