

New Methods for Feynman Integrals

V.A. Smirnov

Nuclear Physics Institute of Moscow State University

- Introduction. Methods of evaluating Feynman integrals
- Reduction to master integrals using IBP relations. Laporta algorithm and its implementations. [FIRE](#)
- Evaluating Feynman integrals by sector decompositions. [FIESTA](#)
- Evaluating Feynman integrals by the method of Mellin–Barnes representation. [MBresolve.m](#)
- A recent application: the three-loop quark static potential

A given Feynman graph $\Gamma \rightarrow$ tensor reduction \rightarrow various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators.

$$F_{\Gamma}(a_1, a_2, \dots) = \int \dots \int \frac{\mathbf{d}^d k_1 \mathbf{d}^d k_2 \dots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \dots}$$

$$d = 4 - 2\epsilon$$

Besides usual propagators, one can have

$$\frac{1}{(v \cdot k + i0)^a}$$

Methods to evaluate Feynman integrals: analytical, numerical, semianalytical . . .

An old **straightforward** analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

The **standard** modern strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as **recurrence relations**.

A general integral of the given family is expressed as a linear combination of some basic (**master**) integrals.

The whole problem of evaluation→

- constructing a reduction procedure
- evaluating master integrals

Solving reduction problems algorithmically:

- ‘Laporta’s algorithm’

[S. Laporta and E. Remiddi’96; S. Laporta’00; T. Gehrman and E. Remiddi’01]

A public version AIR

[C. Anastasiou and A. Lazopoulos’04]

Private versions

[T. Gehrman and E. Remiddi, M. Czakon, Y. Schröder, C. Sturm, P. Marquard and D. Seidel, V. Velizhanin, ...]

- Baikov’s method

- Gröbner bases. Suggested by O.V. Tarasov [O.V. Tarasov’98]

An alternative approach:

[A.V. Smirnov & V.A. Smirnov’05–07;

A.G. Grozin, A.V. Smirnov and V.A. Smirnov’06

A.V. Smirnov, V.A. Smirnov, and M. Steinhauser’08]

FIRE = Feynman Integrals REduction [A.V. Smirnov'08]
(implemented in Mathematica)

<http://www-ttp.particle.uni-karlsruhe.de/~asmirnov>

Sectors

2^n regions labelled by subsets $\nu \subseteq \{1, \dots, n\}$:

$$\sigma_\nu = \{(a_1, \dots, a_n) : a_i > 0 \text{ if } i \in \nu, a_i \leq 0 \text{ if } i \notin \nu\}$$

Natural ordering.

The goal of reduction: to make more non-positive indices.

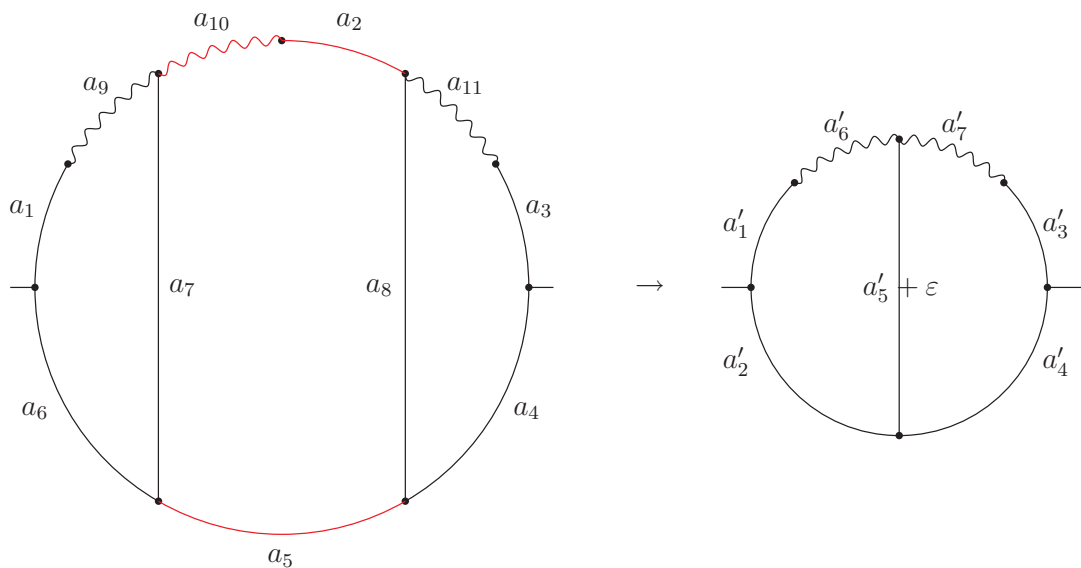
Three different strategies in FIRE.

1. In sectors with a small number of non-positive indices, apply s -bases (generalizations of Gröbner bases). Constructing them automatically by a kind of Buchberger algorithm.

[SBases.m](#)

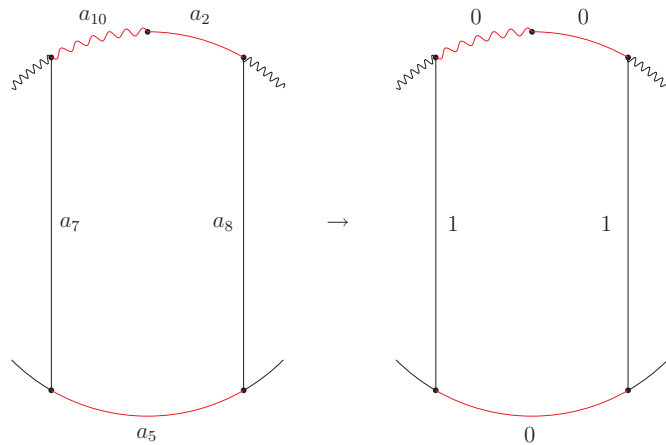
2. In sectors with a large number of non-positive indices, integrate over a loop momentum explicitly and reduce the problem to a family of two-loop integrals where the index of one propagator is, possibly, shifted by ϵ or 2ϵ .

Consider the region $a_2, a_5, a_{10} \leq 0$, $a_7, a_8 > 0$



Apply \mathcal{S} -bases (within FIRE) to such 2loop reduction problems with 7 indices.

Reduce indices $a_2, a_5, a_{10}, a_7, a_8$ to their boundary values,
i.e. $a_2, a_5, a_{10} = 0, a_7, a_8 = 1$



At these values, the transition to the 2loop problem
because very simple (without multiple summations).

3. In ‘intermediate sectors’, the Laporta’s algorithm (implemented within FIRE) is applied. FIRE can be run in a ‘pure Laporta’ mode.

‘Lee ideas’

[R.N. Lee’08]

In each sector one may find a single IBP that works for ‘most’ points in this sector. One might generate less IBPs because the other IBPs are naturally represented as a linear combination of these.

QLink is used to access the QDBM database for storing data on disk from Mathematica.

FLink allows to perform external evaluations by means of the **Fermat** program.

Fermat speeds up `Together` and `GCD`.

<http://www-ttp.particle.uni-karlsruhe.de/~asmirnov>

Methods to evaluate master integrals:

- Feynman/alpha parameters
- Mellin–Barnes representation [V.A. Smirnov'99, J.B Tausk'99]
- method of differential equations [A.V. Kotikov'91, E. Remiddi'97, T. Gehrmann & E. Remiddi'00]

UV, IR and collinear divergences

Regularization.

Dimensional regularization. Formally,

$$d^4k = dk_0 \vec{k} \rightarrow d^d k$$

where $d = 4 - 2\epsilon$

Informally, use alpha parameters

$$\frac{1}{(-k^2 + m^2 - i0)^a} = \frac{i^a}{\Gamma(a)} \int_0^\infty d\alpha \alpha^{a-1} e^{i(k^2 - m^2)\alpha}$$

change the order of integration, take Gauss integrals over the loop momenta

$$\int \mathbf{d}^4 k e^{i(\alpha k^2 - 2q \cdot k)} = -i\pi^2 \alpha^{-2} e^{-iq^2/\alpha}$$

→

$$\int \mathbf{d}^d k e^{i(\alpha k^2 - 2q \cdot k)} = e^{i\pi(1-d/2)/2} \pi^{d/2} \alpha^{-d/2} e^{-iq^2/\alpha}$$

Graph $\Gamma \rightarrow$

$$F_{\Gamma}(a_1, \dots, a_L; d) = \frac{i^{a+h(1-d/2)} \pi^{hd/2}}{\prod_l \Gamma(a_l)} \\ \times \int_0^{\infty} d\alpha_1 \dots \int_0^{\infty} d\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} e^{i\mathcal{V}/\mathcal{U} - i \sum m_i^2 \alpha_i},$$

where h is the number of loops and

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{l \notin T} \alpha_l,$$

$$\mathcal{V} = \sum_{2\text{-trees } T} \prod_{l \notin T} \alpha_l (q^T)^2.$$

$$F_{\Gamma}(q_1, \dots, q_n; d) = \frac{\left(i\pi^{d/2}\right)^h \Gamma(a - hd/2)}{\prod_l \Gamma(a_l)} \\ \times \int_0^{\infty} d\alpha_1 \dots \int_0^{\infty} d\alpha_L \delta\left(\sum \alpha_l - 1\right) \frac{\mathcal{U}^{a-(h+1)d/2} \prod_l \alpha_l^{a_l-1}}{\left(-\mathcal{V} + \mathcal{U} \sum m_l^2 \alpha_l\right)^{a-hd/2}}$$

Sector decompositions.

Hepp sectors

[K. Hepp'66]

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_L$$

sector variables $t_l = \alpha_l/\alpha_{l+1}$, $l = 1, \dots, L - 1$; $t_L = \alpha_L$.

Back:

$$\alpha_l = t_l \dots t_L$$

Speer's sectors

[E. Speer'77]

labelled by the elements of a UV forest \mathcal{F} ,

$$\alpha_l = \prod_{\gamma \in \mathcal{F}: l \in \gamma} t_\gamma$$

For Feynman integrals with the Euclidean external momenta ($(\sum q_i)^2 < 0$ for any subset of external momenta), Speer's sectors are optimal for the resolution of the singularities of the integrand.

Recursively defined sector decompositions

[T. Binoth and G. Heinrich'00]

Primary sectors

$$\alpha_i \leq \alpha_l, \quad l \neq i = 1, 2, \dots, L,$$

with new variables

$$t_i = \begin{cases} \alpha_i/\alpha_l & \text{if } i \neq l \\ \alpha_l & \text{if } i = l \end{cases}$$

The contribution of a primary sector

$$F_l = \int_0^1 \cdots \int_0^1 \left(\prod_{i \neq l} dt_i \right) \frac{\mathcal{U}^{L-(h+1)d/2}}{\mathcal{V}^{L-hd/2}} \Big|_{t_l=1}$$

Next sectors are introduced in similar way.

The goal is to obtain a factorization of \mathcal{U} and \mathcal{V} in final sector variables, i.e. to represent them as products of sector variables in some powers times a positive function.

Strategies that are guaranteed to terminate

[C. Bogner & S. Weinzierl'07]

A, B, C, X

Strategy S

[A.V. Smirnov & M.N. Tentyukov'08]

FIESTA

(Feynman Integral Evaluation by a Sector decomposiTiOn Approach)

<http://www-ttp.particle.uni-karlsruhe.de/~asmirnov>

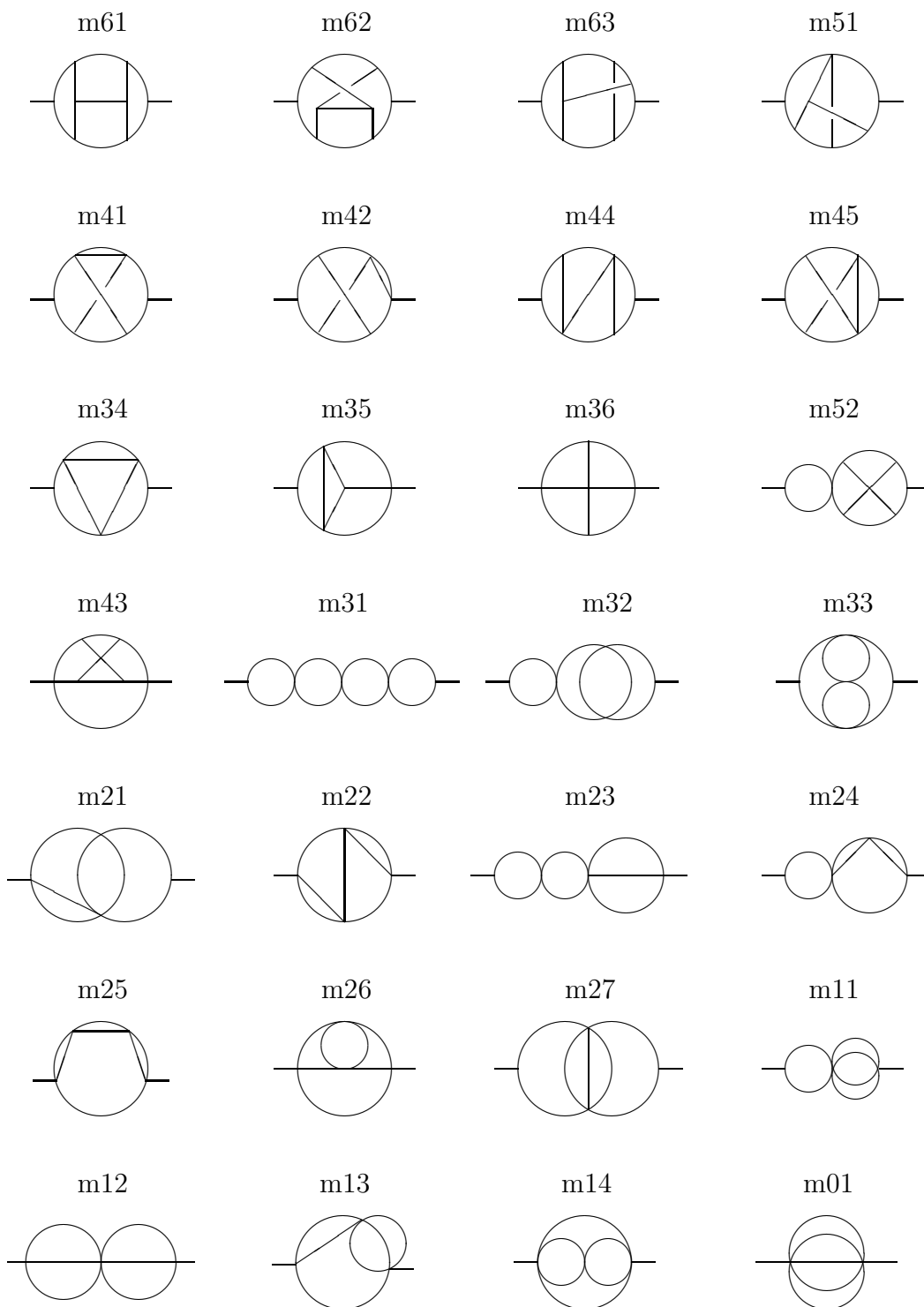
The usage of Speer's sectors within FIESTA.

It turns out that, for Feynman integrals at Euclidean external momenta, Speer' sectors are reproduced within Strategy S

[A.V. Smirnov & V.A. Smirnov'08]

(e.g. for four-loop propagator integrals)

m62: 26304 sectors



Again, as in 3-loop case, "glue-and-cut" relations provide with enough information to express coefficient of expansion over $\epsilon = 2 - D/2$ of all these integrals

MB

Mellin transformation, Mellin integrals as a tool for Feynman integrals: [M.C. Bergère & Y.-M.P. Lam'74]

Evaluating individual Feynman integrals:

[N.I. Ussyukina'75..., A.I. Davydychev'89...,]

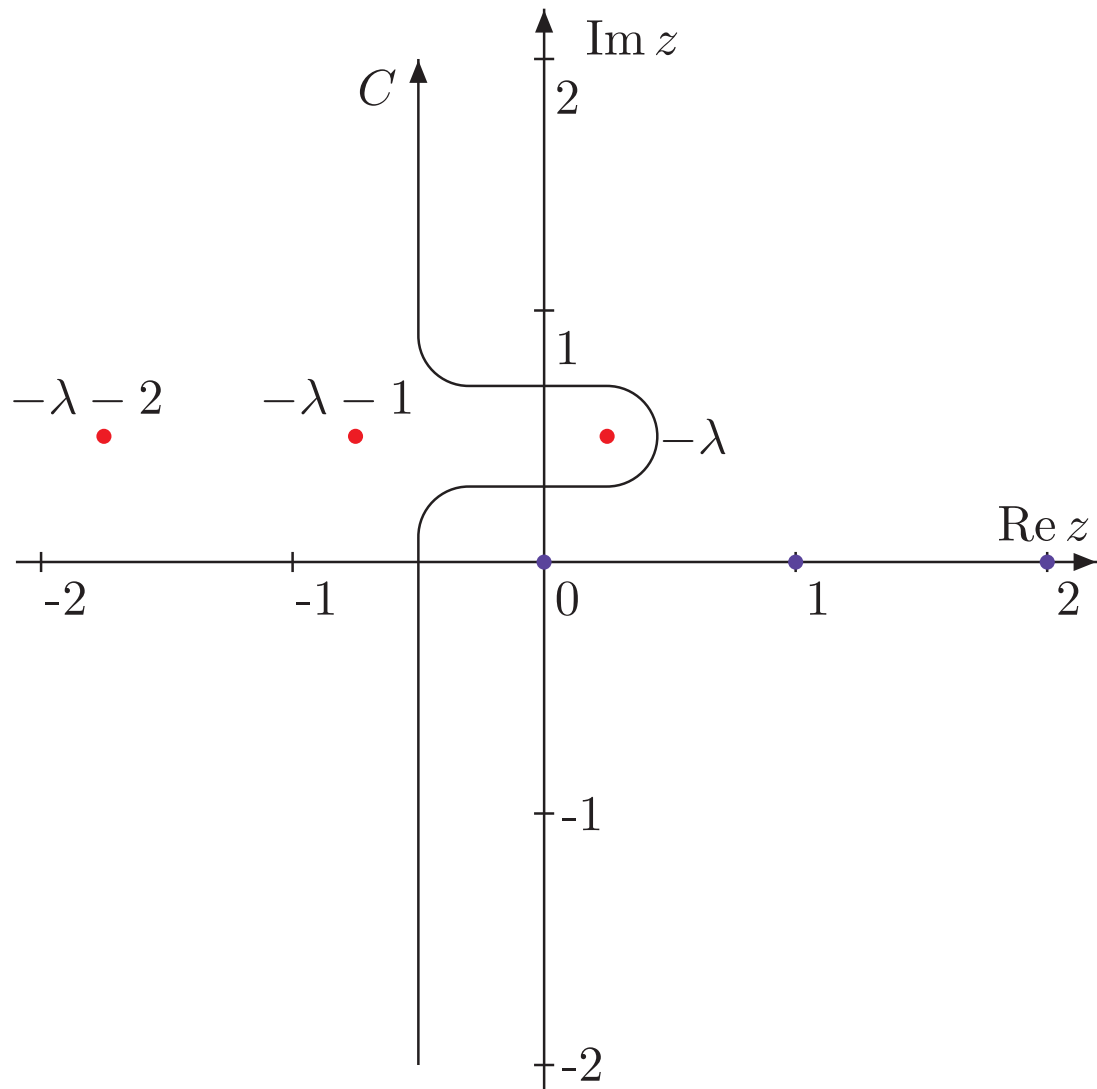
Systematic evaluation of dimensionally regularized Feynman integrals (in particular, systematic resolution of the singularities in ϵ)

[V.A. Smirnov'99, J.B. Tausk'99]

The basic formula:

$$\frac{1}{(X + Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z) \Gamma(-z) .$$

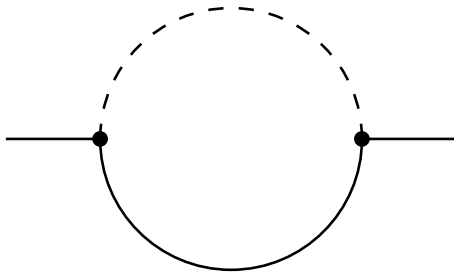
The poles with a $\Gamma(\dots +z)$ dependence are to the left of the contour and the poles with a $\Gamma(\dots -z)$ dependence are to the right



The simplest possibility:

$$\frac{1}{(m^2 - k^2)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}} \Gamma(\lambda + z) \Gamma(-z)$$

Example 1



$$F_\Gamma(q^2, m^2; a_1, a_2, d) = \int \frac{\mathbf{d}^d k}{(m^2 - k^2)^{a_1} (-(q - k)^2)^{a_2}}$$

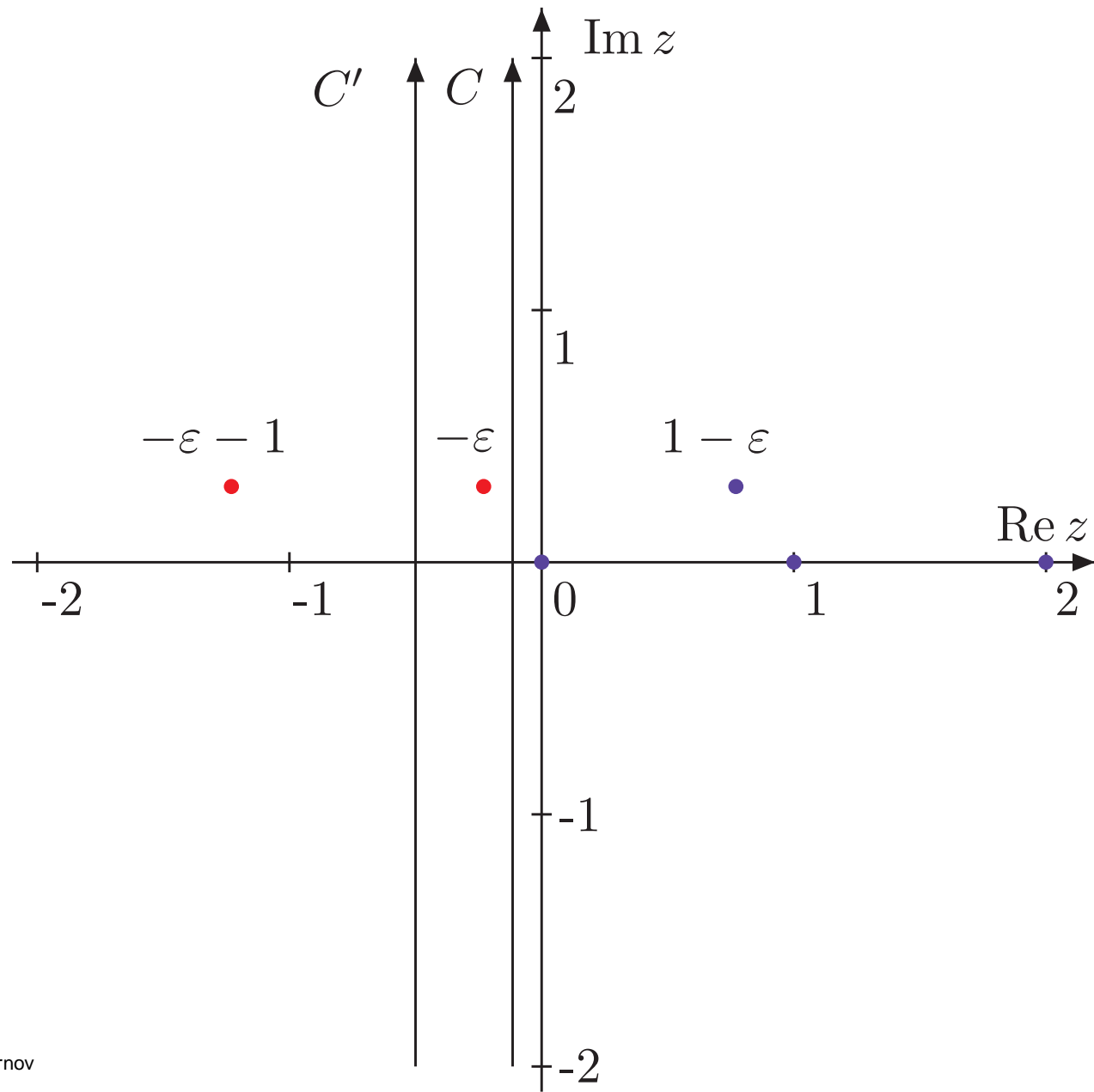
$$\int \frac{\mathbf{d}^d k}{(-k^2)^{a_1} [-(q-k)^2]^{a_2}} = i\pi^{d/2} \frac{G(a_1, a_2)}{(-q^2)^{a_1+a_2+\epsilon-2}} ,$$

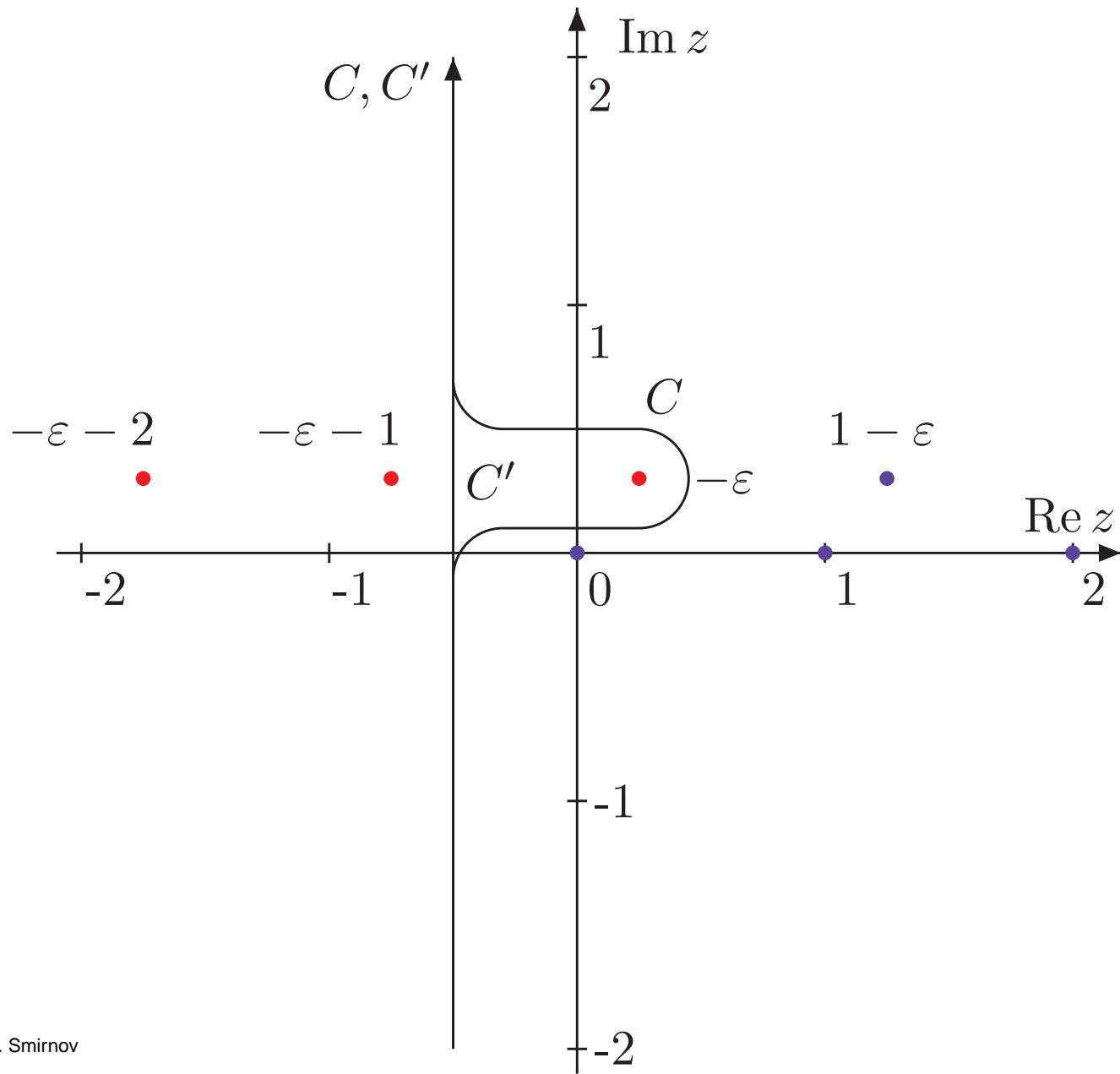
$$G(a_1, a_2) = \frac{\Gamma(a_1 + a_2 + \epsilon - 2)\Gamma(2 - \epsilon - a_1)\Gamma(2 - \epsilon - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4 - a_1 - a_2 - 2\epsilon)}$$

$$\begin{aligned} F_\Gamma(q^2, m^2; a_1, a_2, d) &= \frac{i\pi^{d/2}(-1)^{a_1+a_2}\Gamma(2 - \epsilon - a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \\ &\times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1 + a_2 + \epsilon - 2 + z) \\ &\times \frac{\Gamma(2 - \epsilon - a_1 - z)\Gamma(-z)}{\Gamma(4 - 2\epsilon - a_1 - a_2 - z)} \end{aligned}$$

$$F_{\Gamma}(q^2, m^2; 1, 1, d) = \frac{i\pi^{d/2}\Gamma(1 - \epsilon)}{(-q^2)^{\epsilon}} \\ \times \frac{1}{2\pi i} \int_C \mathbf{d}z \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z)}{\Gamma(2 - 2\epsilon - z)}$$

$\Gamma(\epsilon + z)\Gamma(-z) \rightarrow$ a singularity in ϵ





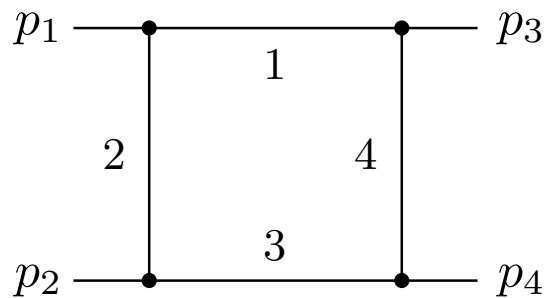
Take a residue at $z = -\epsilon$:

$$i\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon (1 - \epsilon)}$$

and shift the contour:

$$i\pi^2 \frac{1}{2\pi i} \int_{C'} dz \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(z)\Gamma(-z)}{1 - z}$$

Example 2. The massless on-shell box diagram, i.e. with $p_i^2 = 0$, $i = 1, 2, 3, 4$



$$F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) = \int \frac{d^d k}{(k^2)^{a_1} [(k + p_1)^2]^{a_2} [(k + p_1 + p_2)^2]^{a_3} [(k - p_3)^2]^{a_4}},$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

$$\begin{aligned}
& F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) \\
= & (-1)^{a_1} i^{\pi} \pi^{d/2} \frac{\Gamma(a + \epsilon - 2) \Gamma(2 - \epsilon - a_1 - a_2) \Gamma(2 - \epsilon - a_3 - a_4)}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l)} \\
& \times \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1} (1 - \xi_1)^{a_2-1} \xi_2^{a_3-1} (1 - \xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1 - \xi_1)(1 - \xi_2) - i0]^{a+\epsilon-2}} d\xi_1 d\xi_2,
\end{aligned}$$

where $a = a_1 + a_2 + a_3 + a_4$

Apply the basic formula to separate

$-s\xi_1\xi_2$ and $-t(1 - \xi_1)(1 - \xi_2)$ in the denominator

Change the order of integration over z and ξ -parameters,
evaluate parametric integrals in terms of gamma functions

$$\begin{aligned}
F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) &= \frac{(-1)^{a_1} i^{\pi d/2}}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l) (-s)^{a+\epsilon-2}} \\
&\times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z \left(\frac{t}{s}\right)^z \Gamma(a + \epsilon - 2 + z) \Gamma(a_2 + z) \Gamma(a_4 + z) \Gamma(-z) \\
&\times \Gamma(2 - a_1 - a_2 - a_4 - \epsilon - z) \Gamma(2 - a_2 - a_3 - a_4 - \epsilon - z)
\end{aligned}$$

General prescriptions

- Derive a (multiple) MB representation for general powers of the propagators. (The number of MB integrations can be large (more than 10)).
- Use it for checks. Reducing a line to a point \rightarrow tending a_i to zero \rightarrow (usually) taking some residues. A typical situation:
$$\frac{\Gamma(a_2+z)\Gamma(-z)}{\Gamma(a_2)}, \quad a_2 \rightarrow 0$$

Gluing of poles of different nature. Take a (minus) residue at $z_2 = 0$, then set $a_2 = 0$.
- Unambiguous prescriptions for choosing integration contours
- Try to have a minimal number of MB integrations.

- Resolve the singularity structure in ϵ . The goal: to represent a given MB integral as a sum of integrals where a Laurent expansion in ϵ becomes possible. The basic procedure:
take residues and shift contours

Two strategies:

● #1

[V.A. Smirnov'99]

E.g., the product $\Gamma(1+z)\Gamma(-1-\epsilon-z)$ generates a pole of the type $\Gamma(-\epsilon)$.

The general rule: $\Gamma(a+z)\Gamma(b-z)$, where a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$.

Analysis of the integrand. Analysis of integrations over MB variables in various order leads to understanding what 'key' gamma functions (responsible for the generation of poles in ϵ) are.

'Changing the nature' of these gamma functions (i.e. changing rules for the contours).

Taking residues and shifting contours.

● #2

[J.B. Tausk'99, Anastasiou'05, Czakon'05].

Choose a domain of ϵ and $\operatorname{Re} z_i, \dots, \operatorname{Re} w_i$ in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis. Let $\epsilon \rightarrow 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue. For every resulting residue, which involves one integration less, apply a similar procedure, etc.

Two algorithmic descriptions

[C. Anastasiou'05, M. Czakon'05]

The Czakon's version `MB.m` implemented in `Mathematica` is public.

- #1 in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'08]

Let

$$\prod_i \Gamma \left(a_i(\epsilon) + \sum_j b_{ij}(\epsilon) z_j \right)$$

be the numerator of a multiple MB integral. Let ϵ be real.
 ‘Changing the nature’ of the key gamma functions (i.e. changing rules for the contours)

$$\Gamma \left(a_i(\epsilon) + \sum_j b_{ij}(\epsilon) z_j \right) \rightarrow \Gamma^{(1)} \left(a_i(\epsilon) + \sum_j b_{ij}(\epsilon) z_j \right) \rightarrow \dots \Gamma^{(n)} (\dots) \dots$$

instead of $a_i(\epsilon) + \sum_j b_{ij}(\epsilon) \mathbf{Re} z_j > 0$

we have $-1 < a_i(\epsilon) + \sum_j b_{ij}(\epsilon) \mathbf{Re} z_j < 0, \dots,$

$-n - 1 < a_i(\epsilon) + \sum_j b_{ij}(\epsilon) \mathbf{Re} z_j < -n, \dots$

Strategy #2: straight contours in the beginning

Strategy #1: straight contours in the end

Set $\epsilon = 0$

Look for straight contours (i.e. $\text{Re} z_i$) for which gamma functions are changed in a minimal way.

Let $S(x) = [(1 - x)_+]$ where $[...]$ is the integer part of a number and $x_+ = x$ for $x > 0$ and 0 otherwise.

Look for contours for which

$\sum_i S \left(a_i(0) + \sum_j b_{ij}(0) z_j \right)$ is minimal.

With such a choice, identify gamma functions which should be 'changed'.

Take a residue and replace Γ by $\Gamma^{(1)}$ (and, possibly, $\Gamma^{(1)}$ by $\Gamma^{(2)}$ etc.) Proceed iteratively.

[MBresolve.m](#)

<http://www-ttp.particle.uni-karlsruhe.de/~asmirnov>

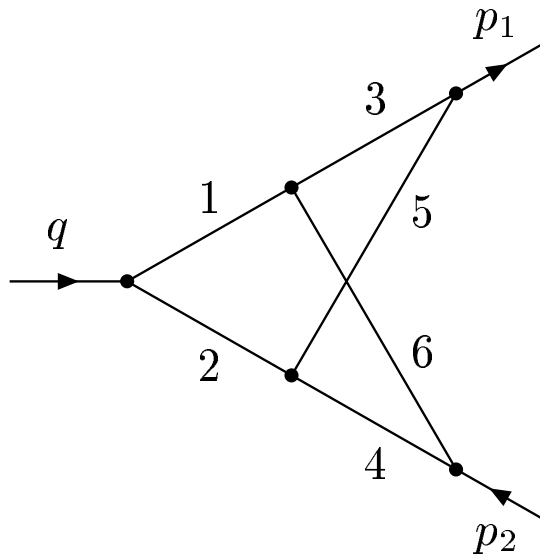
Evaluate MB integrals after expanding in ϵ .

Apply the first and the second **Barnes lemmas**

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z) \\ &= \frac{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} \end{aligned}$$

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{\Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 + z)\Gamma(\lambda_4 - z)\Gamma(\lambda_5 - z)}{\Gamma(\lambda_6 + z)} \\ &= \frac{\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_4)\Gamma(\lambda_3 + \lambda_4)\Gamma(\lambda_1 + \lambda_5)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5)\Gamma(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5)} \\ & \times \frac{\Gamma(\lambda_2 + \lambda_5)\Gamma(\lambda_3 + \lambda_5)}{\Gamma(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)}, \quad \lambda_6 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \end{aligned}$$

Example 3. Non-planar two-loop massless vertex diagram
 with $p_1^2 = p_2^2 = 0$, $Q^2 = -(p_1 - p_2)^2 = 2p_1 \cdot p_2$



$$F_{\Gamma}(Q^2; a_1, \dots, a_6, d) = \int \int \frac{d^d k d^d l}{1 \left[(k+l)^2 - 2p_1 \cdot (k+l) \right]^{a_1} \left[(k+l)^2 - 2p_2 \cdot (k+l) \right]^{a_2} (k^2 - 2p_1 \cdot k)^{a_3} (l^2 - 2p_2 \cdot l)^{a_4} (k^2)^{a_5} (l^2)^{a_6}}$$

Gonsalves'83:

$$\begin{aligned}
 F_{\Gamma}(Q^2; a_1, \dots, a_6, d) &= \frac{(-1)^a \left(i\pi^{d/2}\right)^2 \Gamma(2 - \epsilon - a_{35})\Gamma(2 - \epsilon - a_{46})}{(Q^2)^{a+2\epsilon-4} \prod \Gamma(a_l) \Gamma(4 - 2\epsilon - a_{3456})} \\
 &\times \Gamma(a + 2\epsilon - 4) \int_0^1 d\xi_1 \dots \int_0^1 d\xi_4 \xi_1^{a_3-1} (1 - \xi_1)^{a_5-1} \xi_2^{a_4-1} (1 - \xi_2)^{a_6-1} \\
 &\times \xi_3^{a_1-1} \xi_4^{a_2-1} (1 - \xi_3 - \xi_4)_+^{a_{3456}+\epsilon-3} A(\xi_1, \xi_2, \xi_3, \xi_4)^{4-2\epsilon-a},
 \end{aligned}$$

where

$$A(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_3 \xi_4 + (1 - \xi_3 - \xi_4) [\xi_2 \xi_3 (1 - \xi_1) + \xi_1 \xi_4 (1 - \xi_2)]$$

$$\begin{aligned}
& \frac{\Gamma(a + 2\epsilon - 4)}{[\eta\xi(1 - \xi) + (1 - \eta)(\xi\xi_2(1 - \xi_1) + (1 - \xi)\xi_1(1 - \xi_2))]^{a+2\epsilon-4}} \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z_1 \frac{\Gamma(-z_1) \eta^{z_1} \xi^{z_1} (1 - \xi)^{z_1}}{(1 - \eta)^{a+2\epsilon-4+z_1}} \\
&\quad \times \frac{\Gamma(a + 2\epsilon - 4 + z_1)}{[\xi\xi_2(1 - \xi_1) + (1 - \xi)\xi_1(1 - \xi_2)]^{a+2\epsilon-4+z_1}}
\end{aligned}$$

The last line \rightarrow

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z_2 \frac{\Gamma(a + 2\epsilon - 4 + z_1 + z_2) \Gamma(-z_2) \xi^{z_2} \xi_2^{z_2} (1 - \xi_1)^{z_2}}{(1 - \xi)^{a+2\epsilon-4+z_1+z_2} \xi_1^{a+2\epsilon-4+z_1+z_2} (1 - \xi_2)^{a+2\epsilon-4+z_1+z_2}}$$

$$\begin{aligned}
F_{\Gamma}(Q^2; a_1, \dots, a_6, d) &= \frac{(-1)^a \left(i\pi^{d/2}\right)^2 \Gamma(2 - \epsilon - a_{35})}{(Q^2)^{a+2\epsilon-4} \Gamma(6 - 3\epsilon - a) \prod \Gamma(a_l)} \\
&\times \frac{\Gamma(2 - \epsilon - a_{46})}{\Gamma(4 - 2\epsilon - a_{3456})} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} dz_1 dz_2 \Gamma(a + 2\epsilon - 4 + z_1 + z_2) \\
&\quad \times \Gamma(-z_1) \Gamma(-z_2) \Gamma(a_4 + z_2) \Gamma(a_5 + z_2) \Gamma(a_1 + z_1 + z_2) \\
&\quad \times \frac{\Gamma(2 - \epsilon - a_{12} - z_1) \Gamma(4 - 2\epsilon + a_2 - a - z_2)}{\Gamma(4 - 2\epsilon - a_{1235} - z_1) \Gamma(4 - 2\epsilon - a_{1246} - z_1)} \\
&\quad \times \Gamma(4 - 2\epsilon + a_3 - a - z_1 - z_2) \Gamma(4 - 2\epsilon + a_6 - a - z_1 - z_2) ,
\end{aligned}$$

where $a_{3456} = a_3 + a_4 + a_5 + a_6$, etc.

```

ln[1]:= SetDirectory["c:/diskE/job2008/Bern"];

ln[2]:= << MB/MB.m;
       << MB/MBresolve.m

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

MBresolve 1.0

by Alexander Smirnov

last modified 22 Oct 08

(* 2fold MB representation for the non-planar vertex massless diagram.
The factor  $QQ^{4-a_1-a_2-a_3-a_4-a_5-a_6-2\epsilon}$  is omitted.
 $QQ=-(p_1-p_2)^2$ .
The factor  $(I \Pi^{(d/2)})^2$  is also omitted as usually. *)

ln[4]:= NPMB[a1_, a2_, a3_, a4_, a5_, a6_] := ((-1)^(a1 + a2 + a3 + a4 + a5 + a6) /
(Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[a5] Gamma[a6])
Gamma[2 - epsilon - a3 - a5] Gamma[2 - epsilon - a4 - a6] / Gamma[4 - 2 epsilon - a3 - a4 - a5 - a6] /
Gamma[6 - 3 epsilon - a1 - a2 - a3 - a4 - a5 - a6]
Gamma[a1 + a2 + a3 + a4 + a5 + a6 + 2 epsilon - 4 + z1 + z2] Gamma[-z1] Gamma[-z2]
Gamma[2 - epsilon - a1 - a2 - z1] Gamma[a4 + z2] Gamma[a1 + z1 + z2]
Gamma[4 - 2 epsilon - a1 - a3 - a4 - a5 - a6 - z2] Gamma[4 - 2 epsilon - a1 - a2 - a4 - a5 - a6 - z1 - z2]
Gamma[a5 + z2] Gamma[4 - 2 epsilon - a1 - a2 - a3 - a4 - a5 - z1 - z2] /
Gamma[4 - 2 epsilon - a1 - a2 - a4 - a6 - z1] / Gamma[4 - 2 epsilon - a1 - a2 - a3 - a5 - z1]);

(* The diagram with all powers of the propagators equal
to one. We shall evaluate it in expansion in epsilon up to epsilon^0. *)

ln[5]:= V2 = NPMB[1, 1, 1, 1, 1, 1]

Out[5]= (Gamma[-epsilon]^2 Gamma[-epsilon - z1] Gamma[-z1] Gamma[-1 - 2 epsilon - z2] Gamma[-1 - 2 epsilon - z1 - z2])^2
Gamma[-z2] Gamma[1 + z2]^2 Gamma[1 + z1 + z2] Gamma[2 + 2 epsilon + z1 + z2]) /
(Gamma[-3 epsilon] Gamma[-2 epsilon] Gamma[-2 epsilon - z1]^2)

ln[6]:= V2rules = MBOptimizedRules[V2, epsilon -> 0, {}, {epsilon}]

Out[6]= {{epsilon -> -5/8}, {z1 -> -1/4, z2 -> -1/4}}

ln[7]:= V2cont = MBcontinue[V2, epsilon -> 0, V2rules];

```

Level 1

Taking -residue in $z_2 = -1 - 2 \text{ ep}$

Taking -residue in $z_2 = -1 - 2 \text{ ep} - z_1$

Level 2

Integral {1}

Taking +residue in $z_1 = 2 \text{ ep}$

Integral {2}

Level 3

Integral {1, 1}

4 integral(s) found

In[8]:= **V2select = MBpreselect [MBmerge [V2cont], {ep, 0, 0}]**

In[9]:= **V2exp = Simplify [MBexpand [V2select, Exp [2 ep EulerGamma], {ep, 0, 0}]]**

Out[9]= $\left\{ \text{MBint} \left[\frac{1}{\text{ep}^4} - \frac{\pi^2}{2 \text{ep}^2} - \frac{41 \pi^4}{40} + \frac{55 \text{PolyGamma}[2, 1]}{3 \text{ep}}, \{\{\text{ep} \rightarrow 0\}, \{\}\} \right], \right.$
 $\text{MBint} \left[\frac{1}{4 \text{ep}^2} \text{Gamma}[-z_1]^2 \text{Gamma}[z_1] \text{Gamma}[1 + z_1] \right.$
 $(12 + 12 \text{ep EulerGamma} + 6 \text{ep}^2 \text{EulerGamma}^2 - 7 \text{ep}^2 \pi^2 + 6 \text{ep}^2 \text{PolyGamma}[0, -z_1]^2 +$
 $12 \text{ep}^2 \text{PolyGamma}[0, z_1]^2 - 12 \text{ep}^2 \text{PolyGamma}[0, 1 + z_1]^2 - 24 \text{ep} \text{PolyGamma}[0, z_1]$
 $(1 + \text{ep EulerGamma} + \text{ep PolyGamma}[0, 1 + z_1]) + 12 \text{ep PolyGamma}[0, -z_1]$
 $(3 + 3 \text{ep EulerGamma} - 2 \text{ep PolyGamma}[0, z_1] + 4 \text{ep PolyGamma}[0, 1 + z_1]) -$
 $66 \text{ep}^2 \text{PolyGamma}[1, -z_1] + 12 \text{ep}^2 \text{PolyGamma}[1, z_1] - 12 \text{ep}^2 \text{PolyGamma}[1, 1 + z_1]) ,$
 $\left. \left\{ \{\text{ep} \rightarrow 0\}, \left\{ z_1 \rightarrow -\frac{1}{4} \right\} \right\} \right], \text{MBint} \left[6 \text{Gamma}[-1 - z_2] \text{Gamma}[-1 - z_1 - z_2]^2 \text{Gamma}[-z_2] \right.$
 $\left. \text{Gamma}[1 + z_2]^2 \text{Gamma}[1 + z_1 + z_2] \text{Gamma}[2 + z_1 + z_2], \left\{ \{\text{ep} \rightarrow 0\}, \left\{ z_1 \rightarrow -\frac{1}{4}, z_2 \rightarrow -\frac{1}{4} \right\} \right\} \right] \right\}$

In[10]:= **MBresolve [V2, ep]**

In[11]:= **Simplify [%]**

Out[11]= $\left\{ \text{MBint} \left[\frac{\text{Gamma}[-2 \text{ep}]^4 \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1 + 2 \text{ep}]^2}{\text{Gamma}[-4 \text{ep}]^2}, \{\{\text{ep} \rightarrow 0\}, \{\}\} \right], \right.$
 $\text{MBint} \left[\left(\text{Gamma}[-2 \text{ep}] \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1 + 2 \text{ep}] \text{Gamma}[-\text{ep} - z_1] \text{Gamma}[-z_1]^3 \text{Gamma}[1 + z_1] \right. \right.$
 $\left. \left. \text{Gamma}[-2 \text{ep} + z_1] \right) / \left(\text{Gamma}[-3 \text{ep}] \text{Gamma}[-2 \text{ep} - z_1]^2 \right), \{\{\text{ep} \rightarrow 0\}, \{z_1 \rightarrow -0.859981\}\} \right],$
 $\text{MBint} \left[\frac{1}{\text{Gamma}[-3 \text{ep}]} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[-1 - 2 \text{ep} - z_2] \text{Gamma}[-z_2] \text{Gamma}[1 + \text{ep} + z_2] \right.$
 $\left. \text{Gamma}[1 + 2 \text{ep} + z_2] (-\text{EulerGamma} - \text{PolyGamma}[0, -2 \text{ep}] - 2 \text{PolyGamma}[0, 1 + z_2] + \right.$
 $\left. \text{PolyGamma}[0, 1 + \text{ep} + z_2] + \text{PolyGamma}[0, 1 + 2 \text{ep} + z_2]) , \{\{\text{ep} \rightarrow 0\}, \{z_2 \rightarrow -0.859981\}\} \right],$
 $\text{MBint} \left[\left(\text{Gamma}[-\text{ep}]^2 \text{Gamma}[-\text{ep} - z_1] \text{Gamma}[-z_1] \text{Gamma}[-1 - 2 \text{ep} - z_2] \text{Gamma}[-1 - 2 \text{ep} - z_1 - z_2]^2 \right. \right.$
 $\left. \left. \text{Gamma}[-z_2] \text{Gamma}[1 + z_2]^2 \text{Gamma}[1 + z_1 + z_2] \text{Gamma}[2 + 2 \text{ep} + z_1 + z_2] \right) / \right.$
 $\left. \left(\text{Gamma}[-3 \text{ep}] \text{Gamma}[-2 \text{ep}] \text{Gamma}[-2 \text{ep} - z_1]^2 \right), \left\{ \{\text{ep} \rightarrow 0\}, \{z_1 \rightarrow -0.72274, z_2 \rightarrow -0.274294\} \right\} \right] \right\}$

In[12]:= **V2select = MBpreselect [MBmerge [%], {ep, 0, 0}]**

In[13]:= **V2expS = Simplify[MBexpand[%, Exp[2 ep EulerGamma], {ep, 0, 0}]]**

$$\text{Out[13]= } \left\{ \text{MBint} \left[\frac{1}{\text{ep}^4} - \frac{\pi^2}{2 \text{ep}^2} - \frac{41 \pi^4}{40} + \frac{55 \text{PolyGamma}[2, 1]}{3 \text{ep}}, \{ \{\text{ep} \rightarrow 0\}, \{\} \} \right], \right. \\ \text{MBint} \left[\frac{1}{8 \text{ep}^2} \text{Gamma}[-z1]^2 \text{Gamma}[z1] \text{Gamma}[1+z1] \right. \\ \left. (12 + 12 \text{ep EulerGamma} + 6 \text{ep}^2 \text{EulerGamma}^2 + \text{ep}^2 \pi^2 + 54 \text{ep}^2 \text{PolyGamma}[0, -z1]^2 - \right. \\ \left. 24 \text{ep} (1 + \text{ep EulerGamma}) \text{PolyGamma}[0, z1] + 24 \text{ep}^2 \text{PolyGamma}[0, z1]^2 + \right. \\ \left. 36 \text{ep} \text{PolyGamma}[0, -z1] (1 + \text{ep EulerGamma} - 2 \text{ep} \text{PolyGamma}[0, z1]) - \right. \\ \left. 42 \text{ep}^2 \text{PolyGamma}[1, -z1] + 24 \text{ep}^2 \text{PolyGamma}[1, z1]), \{ \{\text{ep} \rightarrow 0\}, \{z1 \rightarrow -0.859981\} \} \right], \\ \text{MBint} \left[\frac{1}{8 \text{ep}^2} 3 \text{Gamma}[-1-z2] \text{Gamma}[-z2] \text{Gamma}[1+z2]^2 \right. \\ \left. (4 + 4 \text{ep EulerGamma} + 2 \text{ep}^2 \text{EulerGamma}^2 - 5 \text{ep}^2 \pi^2 + 8 \text{ep}^2 \text{PolyGamma}[0, -1-z2]^2 + \right. \\ \left. 12 \text{ep} (1 + \text{ep EulerGamma}) \text{PolyGamma}[0, 1+z2] + 18 \text{ep}^2 \text{PolyGamma}[0, 1+z2]^2 - \right. \\ \left. 8 \text{ep} \text{PolyGamma}[0, -1-z2] (1 + \text{ep EulerGamma} + 3 \text{ep} \text{PolyGamma}[0, 1+z2]) + \right. \\ \left. 8 \text{ep}^2 \text{PolyGamma}[1, -1-z2] - 14 \text{ep}^2 \text{PolyGamma}[1, 1+z2]), \{ \{\text{ep} \rightarrow 0\}, \{z2 \rightarrow -0.859981\} \} \right], \\ \text{MBint} \left[6 \text{Gamma}[-1-z2] \text{Gamma}[-1-z1-z2]^2 \text{Gamma}[-z2] \text{Gamma}[1+z2]^2 \text{Gamma}[1+z1+z2] \right. \\ \left. \text{Gamma}[2+z1+z2], \{ \{\text{ep} \rightarrow 0\}, \{z1 \rightarrow -0.72274, z2 \rightarrow -0.274294\} \} \right] \left. \right\}$$

In[14]:= **res1 = V2exp[[1]][[1]]**

$$\text{Out[14]= } \frac{1}{\text{ep}^4} - \frac{\pi^2}{2 \text{ep}^2} - \frac{41 \pi^4}{40} + \frac{55 \text{PolyGamma}[2, 1]}{3 \text{ep}}$$

In[15]:= **V2exp[[3]]**

$$\text{Out[15]= } \text{MBint} \left[6 \text{Gamma}[-1-z2] \text{Gamma}[-1-z1-z2]^2 \text{Gamma}[-z2] \text{Gamma}[1+z2]^2 \right. \\ \left. \text{Gamma}[1+z1+z2] \text{Gamma}[2+z1+z2], \{ \{\text{ep} \rightarrow 0\}, \{z1 \rightarrow -\frac{1}{4}, z2 \rightarrow -\frac{1}{4}\} \} \right]$$

In[16]:= **Barnes1[V2exp[[3]], z1]**

$$\text{Out[16]= } \text{MBint} \left[\pi^2 \text{Gamma}[-1-z2] \text{Gamma}[-z2] \text{Gamma}[1+z2]^2, \{ \{\text{ep} \rightarrow 0\}, \{z2 \rightarrow -\frac{1}{4}\} \} \right]$$

In[17]:= **res2 = Barnes1[%, z2][[1]]**

$$\text{Out[17]= } -\frac{\pi^4}{6}$$

In[18]:= **V2exp[[2]]**

$$\text{Out[18]= } \text{MBint} \left[\frac{1}{4 \text{ep}^2} \right. \\ \left. \text{Gamma}[-z1]^2 \text{Gamma}[z1] \text{Gamma}[1+z1] (12 + 12 \text{ep EulerGamma} + 6 \text{ep}^2 \text{EulerGamma}^2 - 7 \text{ep}^2 \pi^2 + \right. \\ \left. 6 \text{ep}^2 \text{PolyGamma}[0, -z1]^2 + 12 \text{ep}^2 \text{PolyGamma}[0, z1]^2 - 12 \text{ep}^2 \text{PolyGamma}[0, 1+z1]^2 - \right. \\ \left. 24 \text{ep} \text{PolyGamma}[0, z1] (1 + \text{ep EulerGamma} + \text{ep} \text{PolyGamma}[0, 1+z1]) + 12 \text{ep} \right. \\ \left. \text{PolyGamma}[0, -z1] (3 + 3 \text{ep EulerGamma} - 2 \text{ep} \text{PolyGamma}[0, z1] + 4 \text{ep} \text{PolyGamma}[0, 1+z1]) - \right. \\ \left. 66 \text{ep}^2 \text{PolyGamma}[1, -z1] + 12 \text{ep}^2 \text{PolyGamma}[1, z1] - \right. \\ \left. 12 \text{ep}^2 \text{PolyGamma}[1, 1+z1]), \{ \{\text{ep} \rightarrow 0\}, \{z1 \rightarrow -\frac{1}{4}\} \} \right]$$

In[19]:= **CoeffEps[X_, n_] := (X /. X[[1]] → Simplify[Coefficient[X[[1]], ep, n]);**

In[20]:= **CoeffEps**[**V2exp**[[2]], -2]

Out[20]= $\text{MBint}\left[3 \Gamma[-z_1]^2 \Gamma[z_1] \Gamma[1+z_1], \left\{\{ep \rightarrow 0\}, \left\{z_1 \rightarrow -\frac{1}{4}\right\}\right\}\right]$

In[21]:= **res32** = **Barnes1**[**CoeffEps**[**V2exp**[[2]], -2], **z1**][[1]]

Out[21]= $-\frac{\pi^2}{2}$

In[22]:= **CoeffEps**[**V2exp**[[2]], -1]

Out[22]= $\text{MBint}\left[3 \Gamma[-z_1]^2 \Gamma[z_1] \Gamma[1+z_1] \left(\text{EulerGamma} + 3 \text{PolyGamma}[0, -z_1] - 2 \text{PolyGamma}[0, z_1]\right), \left\{\{ep \rightarrow 0\}, \left\{z_1 \rightarrow -\frac{1}{4}\right\}\right\}\right]$

In[23]:= **res31** = **9 Zeta**[3];

In[24]:= **res31** // **N**

Out[24]= 10.8185

In[25]:= **NIntegrate**[**CoeffEps**[**V2exp**[[2]], -1][[1]] / (2 Pi) /. {**z1** $\rightarrow -\frac{1}{4} + \text{I} * \text{y1}$ }, {**y1**, -Infinity, Infinity}]

Out[25]= 10.8185 + 2.13163 $\times 10^{-14}$ i

In[26]:= **CoeffEps**[**V2exp**[[2]], 0]

Out[26]= $\text{MBint}\left[\frac{1}{4} \Gamma[-z_1]^2 \Gamma[z_1] \Gamma[1+z_1] \left(6 \text{EulerGamma}^2 - 7 \pi^2 + 6 \text{PolyGamma}[0, -z_1]^2 + 12 \text{PolyGamma}[0, z_1]^2 - 12 \text{PolyGamma}[0, 1+z_1]^2 - 24 \text{PolyGamma}[0, z_1] (\text{EulerGamma} + \text{PolyGamma}[0, 1+z_1]) + 12 \text{PolyGamma}[0, -z_1] (3 \text{EulerGamma} - 2 \text{PolyGamma}[0, z_1] + 4 \text{PolyGamma}[0, 1+z_1]) - 66 \text{PolyGamma}[1, -z_1] + 12 \text{PolyGamma}[1, z_1] - 12 \text{PolyGamma}[1, 1+z_1]\right), \left\{\{ep \rightarrow 0\}, \left\{z_1 \rightarrow -\frac{1}{4}\right\}\right\}\right]$

In[27]:= **res30** = $\frac{7 \pi^4}{10}$;

In[28]:= **res30** // **N**

Out[28]= 68.1864

In[29]:= **NIntegrate**[**CoeffEps**[**V2exp**[[2]], 0][[1]] / (2 Pi) /. {**z1** $\rightarrow -\frac{1}{4} + \text{I} * \text{y1}$ }, {**y1**, -Infinity, Infinity}]

Out[29]= 68.1864 + 0. i

(* **result** *)

In[30]:= **FullSimplify**[**res1** + **res2** + **res32** / **ep**^2 + **res31** / **ep** + **res30**]

Out[30]= $\frac{1}{ep^4} - \frac{\pi^2}{ep^2} - \frac{59 \pi^4}{120} - \frac{83 \text{Zeta}[3]}{3 ep}$

```

In[1]:= SetDirectory["c:/diskE/job2008/Bern"];

In[2]:= << MB/MB.m;
        << MB/MBresolve.m

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

MBresolve 1.0

by Alexander Smirnov

last modified 22 Oct 08

In[4]:= K2[a1_, a2_, a3_, a4_, a5_, a6_, a7_, a8_] :=
( xz1 Gamma[-z1] Gamma[a2+z1] Gamma[2-a5-a6-a7-ep-z1-z2]
  Gamma[2-a1-a2-a8-ep+z2] Gamma[2-a4-a5-a7-ep-z1-z3]
  Gamma[2-a2-a3-a8-ep+z3] Gamma[a7+z1-z4] Gamma[-2+a4+a5+a6+a7+ep+z1-z4]
  Gamma[a8-z2-z3-z4] Gamma[-z1-z2-z3-z4] Gamma[-2+a1+a2+a3+a8+ep+z4]
  Gamma[z2+z4] Gamma[z3+z4] Gamma[a5+z1+z2+z3+z4] ) /
(Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4-a4-a5-a6-a7-2ep]
 Gamma[4-a1-a2-a3-a8-2ep+z1-z4]
 Gamma[a8-z1-z2-z3-z4] Gamma[a3+z2+z4] Gamma[a1+z3+z4])

In[5]:= B2 = K2[1, 1, 1, 1, 1, 1, 1, -1]

Out[5]= ( xz1 Gamma[-z1] Gamma[1+z1] Gamma[-1-ep-z1-z2]
  Gamma[1-ep+z2] Gamma[-1-ep-z1-z3] Gamma[1-ep+z3] Gamma[1+z1-z4]
  Gamma[2+ep+z1-z4] Gamma[-1-z2-z3-z4] Gamma[-z1-z2-z3-z4]
  Gamma[ep+z4] Gamma[z2+z4] Gamma[z3+z4] Gamma[1+z1+z2+z3+z4] ) /
(Gamma[-2ep] Gamma[2-2ep+z1-z4] Gamma[-1-z1-z2-z3-z4]
 Gamma[1+z2+z4] Gamma[1+z3+z4])

In[6]:= MBOptimizedRules[B2, ep -> 0, {}, {ep}]

MBrules::norules : no rules could be found to regulate this integral

Out[6]= {}

In[7]:= MBresolve[B2, ep]

CREATING RESIDUES LIST0.8594 seconds
FAILED TO RESOLVE

Out[7]= False

In[8]:= B2 = K2[1, 1, 1, 1, 1, 1, 1, -1+y];

In[9]:= rul = MBOptimizedRules[B2, y -> 0, {}, {y, ep}]

Out[9]= { { y -> 55/128, ep -> -107/384 }, { z1 -> 1/6, z2 -> -73/96, z3 -> -295/384, z4 -> 19/24 } }

In[10]:= con1 = MBcontinue[B2, y -> 0, rul];

```

Level 1

Taking -residue in $z_4 = -1 + y - z_2 - z_3$

Level 2

Integral {1}

Taking +residue in $z_1 = -y$

Taking -residue in $z_2 = -1 + y$

Taking -residue in $z_3 = -1 + y$

Level 3

Integral {1, 1}

Integral {1, 2}

Taking +residue in $z_1 = -y$

Taking -residue in $z_3 = -1 + y$

Integral {1, 3}

Taking +residue in $z_1 = -y$

Level 4

Integral {1, 2, 1}

Integral {1, 2, 2}

Taking +residue in $z_1 = -y$

Integral {1, 3, 1}

Level 5

Integral {1, 2, 2, 1}

9 integral(s) found

```
In[11]:= exp1 = MBexpand[con1, 1, {y, 0, 0}] // MBmerge;
```

```
In[12]:= con2 = Table[MBcontinue[exp1[[i, 1]], ep -> 0, exp1[[i, 2]]], {i, Length[exp1]}] // MBmerge;
```

In[13]:= **exp2 = MBexpand[con2, Exp[2 ep EulerGamma], {ep, 0, 0}] // MBmerge**

Out[13]=
$$\left\{ \text{MBint} \left[\frac{1}{1440 \text{ep}^4} (3240 - 2820 \text{ep}^2 \pi^2 + 31 \text{ep}^4 \pi^4 + 9720 \text{ep}^3 \text{PolyGamma}[2, 1] - 480 \text{ep} \text{Log}[x] (6 - \text{ep}^2 \pi^2 + 14 \text{ep}^3 \text{PolyGamma}[2, 1])) \right], \{y \rightarrow 0, \text{ep} \rightarrow 0\}, \{\} \right\},$$

$$\text{MBint} \left[-\frac{1}{\text{ep}} 2 x^{z1} \text{Gamma}[-z1]^2 \text{Gamma}[z1] \text{Gamma}[1+z1] \right.$$

$$\left. (\text{Gamma}[-z1] \text{Gamma}[1+z1] (-1 + \text{ep} \text{EulerGamma} + 4 \text{ep} \text{PolyGamma}[0, -z1] - 2 \text{ep} \text{PolyGamma}[0, z1] - \text{ep} \text{PolyGamma}[0, 1+z1])) + \text{Gamma}[1-z1] \text{Gamma}[z1] (3 - \text{ep} \text{EulerGamma} - 8 \text{ep} \text{PolyGamma}[0, -z1] + 2 \text{ep} \text{PolyGamma}[0, z1] + 5 \text{ep} \text{PolyGamma}[0, 1+z1]) \right),$$

$$\{y \rightarrow 0, \text{ep} \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{6} \right\} \right\}, \text{MBint} \left[\frac{1}{\text{ep}} \text{Gamma}[-1-z2]^2 \text{Gamma}[1+z2]^2 \right.$$

$$\left. (-1 - 2 \text{ep} \text{EulerGamma} + \text{ep} \text{PolyGamma}[0, -1-z2] - 3 \text{ep} \text{PolyGamma}[0, 2+z2]) \right),$$

$$\{y \rightarrow 0, \text{ep} \rightarrow 0\}, \left\{ z2 \rightarrow -\frac{73}{96} \right\} \right\},$$

$$\text{MBint} \left[\frac{1}{\text{ep}} \text{Gamma}[-1-z3]^2 \text{Gamma}[1+z3]^2 \right.$$

$$\left. (-1 - 2 \text{ep} \text{EulerGamma} + \text{ep} \text{PolyGamma}[0, -1-z3] - 3 \text{ep} \text{PolyGamma}[0, 2+z3]) \right),$$

$$\{y \rightarrow 0, \text{ep} \rightarrow 0\}, \left\{ z3 \rightarrow -\frac{295}{384} \right\} \right\}, \text{MBint} \left[\frac{1}{\text{Gamma}[2+z1+z2]} \right.$$

$$\left. 2 x^{z1} \text{Gamma}[-z1] \text{Gamma}[1+z1] \text{Gamma}[-1-z2] \text{Gamma}[1+z2] \text{Gamma}[1+z1+z2] \right.$$

$$\left. (\text{Gamma}[-z1] \text{Gamma}[1+z1] \text{Gamma}[-z1-z2] \text{Gamma}[1+z1+z2] + \text{Gamma}[1-z1] \text{Gamma}[z1] \text{Gamma}[-1-z1-z2] \text{Gamma}[2+z1+z2]) \right),$$

$$\{y \rightarrow 0, \text{ep} \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{6}, z2 \rightarrow -\frac{73}{96} \right\} \right\}, \text{MBint} \left[\frac{1}{\text{Gamma}[2+z1+z3]} \right.$$

$$\left. 2 x^{z1} \text{Gamma}[-z1] \text{Gamma}[1+z1] \text{Gamma}[-1-z3] \text{Gamma}[1+z3] \text{Gamma}[1+z1+z3] \right.$$

$$\left. (\text{Gamma}[-z1] \text{Gamma}[1+z1] \text{Gamma}[-z1-z3] \text{Gamma}[1+z1+z3] + \text{Gamma}[1-z1] \text{Gamma}[z1] \text{Gamma}[-1-z1-z3] \text{Gamma}[2+z1+z3]) \right),$$

$$\{y \rightarrow 0, \text{ep} \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{6}, z3 \rightarrow -\frac{295}{384} \right\} \right\}$$

(*****)

In[14]:= **B2 = K2[1, 1, 1, 1, 1, 1, 1, -1 + y]**

Out[14]=
$$\left(x^{z1} \text{Gamma}[-z1] \text{Gamma}[1+z1] \text{Gamma}[-1-\text{ep}-z1-z2] \right.$$

$$\text{Gamma}[1-\text{ep}-y+z2] \text{Gamma}[-1-\text{ep}-z1-z3] \text{Gamma}[1-\text{ep}-y+z3] \text{Gamma}[1+z1-z4]$$

$$\text{Gamma}[2+\text{ep}+z1-z4] \text{Gamma}[-1+y-z2-z3-z4] \text{Gamma}[-z1-z2-z3-z4]$$

$$\left. \text{Gamma}[\text{ep}+y+z4] \text{Gamma}[z2+z4] \text{Gamma}[z3+z4] \text{Gamma}[1+z1+z2+z3+z4] \right) /$$

$$\left(\text{Gamma}[-2 \text{ep}] \text{Gamma}[2-2 \text{ep}-y+z1-z4] \text{Gamma}[-1+y-z1-z2-z3-z4] \right.$$

$$\left. \text{Gamma}[1+z2+z4] \text{Gamma}[1+z3+z4] \right)$$

In[15]:= **res1 = MBresolve[B2, {ep → 0, y → 0}]**

In[16]:= **res2 = MBpreselect[res1, {y, 0, 0}]**

In[17]:= **res3 = MBexpand[res2, 1, {y, 0, 0}]**

In[18]:= **res4 = MBpreselect[res3, {ep, 0, 0}]**

In[19]:= `res5 = Simplify[MBexpand[res4, Exp[2 ep EulerGamma], {ep, 0, 0}]]`

Out[19]=
$$\left\{ \text{MBint} \left[\frac{5}{2 \text{ep}^4} - \frac{19 \pi^2}{12 \text{ep}^2} - \frac{13 \pi^4}{720} - \pi^2 \text{Log}[x]^2 + \frac{2 \text{Log}[x]^3}{3 \text{ep}} - \frac{\text{Log}[x]^4}{3} + \frac{47 \text{PolyGamma}[2, 1]}{6 \text{ep}} - \frac{\text{Log}[x] (6 - 7 \text{ep}^2 \pi^2 + 20 \text{ep}^3 \text{PolyGamma}[2, 1])}{3 \text{ep}^3}, \{\{\text{ep} \rightarrow 0\}, \{\}\}\right], \right.$$

$$\text{MBint} \left[\frac{1}{\text{ep}} 2 x^{-1+z^4} \text{Gamma}[1-z^4]^2 \text{Gamma}[2-z^4] \text{Gamma}[-1+z^4]^2 \text{Gamma}[z^4] \right.$$

$$\left. (-1 + \text{ep EulerGamma} + 2 \text{ep PolyGamma}[0, 1-z^4] - \text{ep PolyGamma}[0, z^4]), \{\{\text{ep} \rightarrow 0\}, \{z^4 \rightarrow 1.18859\}\}\right], \text{MBint} \left[\frac{1}{\text{ep}} 2 x^{-1+z^4} \text{Gamma}[1-z^4]^2 \text{Gamma}[2-z^4] \text{Gamma}[-1+z^4]^2 \right.$$

$$\left. \text{Gamma}[z^4] (-1 + \text{ep EulerGamma} + 2 \text{ep PolyGamma}[0, 1-z^4] - \text{ep PolyGamma}[0, z^4]), \{\{\text{ep} \rightarrow 0\}, \{z^4 \rightarrow 1.19354\}\}\right],$$

$$\text{MBint} \left[-\frac{1}{\text{ep}} 2 x^{-1+z^4} \text{Gamma}[1-z^4]^3 \text{Gamma}[-1+z^4] \text{Gamma}[z^4]^2 (-1 + \text{ep EulerGamma} + \right.$$

$$\left. 4 \text{ep PolyGamma}[0, 1-z^4] - 2 \text{ep PolyGamma}[0, -1+z^4] - \text{ep PolyGamma}[0, z^4]), \{\{\text{ep} \rightarrow 0\}, \{z^4 \rightarrow 0.649289\}\}\right], \text{MBint} \left[\frac{1}{\text{Gamma}[1+z^3+z^4]} 2 x^{-1+z^4} \text{Gamma}[-1-z^3] \right.$$

$$\left. \text{Gamma}[1+z^3] \text{Gamma}[1-z^4]^2 \text{Gamma}[1-z^3-z^4] \text{Gamma}[z^4]^2 \text{Gamma}[z^3+z^4]^2, \{\{\text{ep} \rightarrow 0\}, \{z^3 \rightarrow -0.965164, z^4 \rightarrow 0.969365\}\}\right], \text{MBint} \left[\frac{1}{\text{Gamma}[1+z^2+z^4]} \right.$$

$$\left. 2 x^{-1+z^4} \text{Gamma}[-1-z^2] \text{Gamma}[1+z^2] \text{Gamma}[1-z^4]^2 \text{Gamma}[1-z^2-z^4] \text{Gamma}[z^4]^2 \text{Gamma}[z^2+z^4]^2, \{\{\text{ep} \rightarrow 0\}, \{z^2 \rightarrow -0.046298, z^4 \rightarrow 0.940037\}\}\right],$$

$$\text{MBint} \left[-\frac{120 + 180 \text{ep}^2 \pi^2 + 173 \text{ep}^4 \pi^4 + 520 \text{ep}^3 \text{PolyGamma}[2, 1]}{480 \text{ep}^4}, \{\{\text{ep} \rightarrow 0\}, \{\}\}\right],$$

$$\text{MBint} \left[-\frac{1}{\text{ep}} \text{Gamma}[1-z^4]^2 \text{Gamma}[-1+z^4]^2 (1 + 2 \text{ep EulerGamma} - 2 \text{ep PolyGamma}[0, 1-z^4] + \right.$$

$$\left. 3 \text{ep PolyGamma}[0, 2-z^4] + \text{ep PolyGamma}[0, -1+z^4]), \{\{\text{ep} \rightarrow 0\}, \{z^4 \rightarrow 1.19354\}\}\right],$$

$$\text{MBint} \left[-\frac{1}{\text{ep}} \text{Gamma}[1-z^4]^2 \text{Gamma}[-1+z^4]^2 (1 + 2 \text{ep EulerGamma} - 2 \text{ep PolyGamma}[0, 1-z^4] + \right.$$

$$\left. 3 \text{ep PolyGamma}[0, 2-z^4] + \text{ep PolyGamma}[0, -1+z^4]), \{\{\text{ep} \rightarrow 0\}, \{z^4 \rightarrow 1.19354\}\}\right],$$

$$\text{MBint} \left[-\frac{1}{\text{ep}} 2 x^{z^1} \text{Gamma}[1-z^1] \text{Gamma}[-z^1]^2 \text{Gamma}[z^1]^2 \text{Gamma}[1+z^1] \right.$$

$$\left. (1 + \text{ep EulerGamma} - 2 \text{ep PolyGamma}[0, -z^1] + 3 \text{ep PolyGamma}[0, 1+z^1]), \{\{\text{ep} \rightarrow 0\}, \{z^1 \rightarrow -0.377479\}\}\right], \text{MBint} \left[2 x^{z^1} \text{Gamma}[1-z^1] \text{Gamma}[-z^1] \text{Gamma}[z^1] \right.$$

$$\left. \text{Gamma}[1+z^1] \text{Gamma}[1-z^4] \text{Gamma}[1+z^1-z^4] \text{Gamma}[-1+z^4] \text{Gamma}[-1-z^1+z^4], \{\{\text{ep} \rightarrow 0\}, \{z^1 \rightarrow -0.731822, z^4 \rightarrow 0.76808\}\}\right],$$

$$\text{MBint} \left[2 x^{z^1} \text{Gamma}[1-z^1] \text{Gamma}[-z^1] \text{Gamma}[z^1] \text{Gamma}[1+z^1] \text{Gamma}[1-z^4] \text{Gamma}[1+z^1-z^4] \right.$$

$$\left. \text{Gamma}[-1+z^4] \text{Gamma}[-1-z^1+z^4], \{\{\text{ep} \rightarrow 0\}, \{z^1 \rightarrow -0.547767, z^4 \rightarrow 0.832684\}\}\right] \left. \right\}$$

The static colour-singlet potential

$$V(|\mathbf{q}|) = -\frac{4\pi C_F \alpha_s(|\mathbf{q}|)}{q^2} \left[1 + \frac{\alpha_s(|\mathbf{q}|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\mathbf{q}|)}{4\pi} \right)^2 a_2 + \left(\frac{\alpha_s(|\mathbf{q}|)}{4\pi} \right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln \frac{\mu^2}{q^2} \right) + \dots \right]$$

with $C_A = N$ and $C_F = (N^2 - 1)/(2N)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $SU(N)$ colour gauge group, respectively, $T_F = 1/2$ is the index of the fundamental representation, and n_l is the number of light-quark flavours.

One loop

[W. Fischler '77; A. Billoire '80]

$$a_1 = \frac{31}{9}C_A - \frac{20}{9}T_F n_l$$

Two loops

[M. Peter'97; Y. Schröder'99]

$$a_2 = \left[\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3) \right] C_A^2 - \left[\frac{1798}{81} + \frac{56}{3}\zeta(3) \right] C_A T_F n_l \\ - \left[\frac{55}{3} - 16\zeta(3) \right] C_F T_F n_l + \left(\frac{20}{9} T_F n_l \right)^2$$

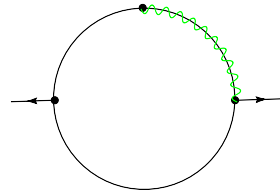
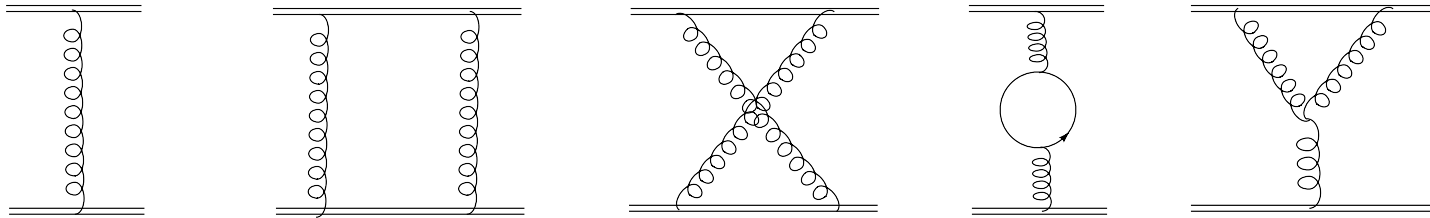
$1/m_q$ -correction

[B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser'02]

Three loops: $a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)} = ?$

a_3 is one of a few missing NNNLO ingredients of the non-relativistic dynamics near threshold.

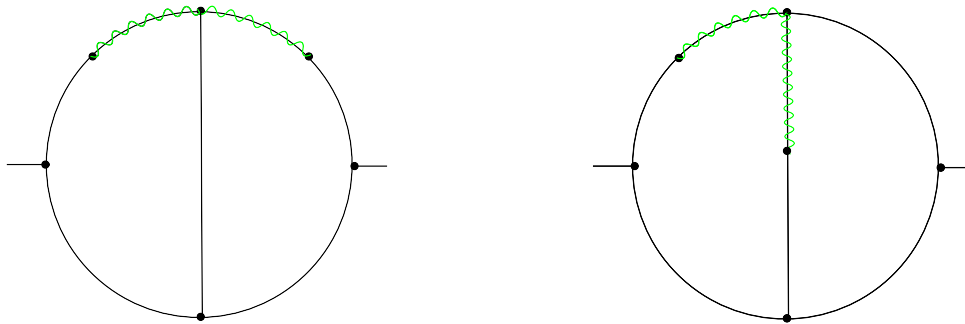
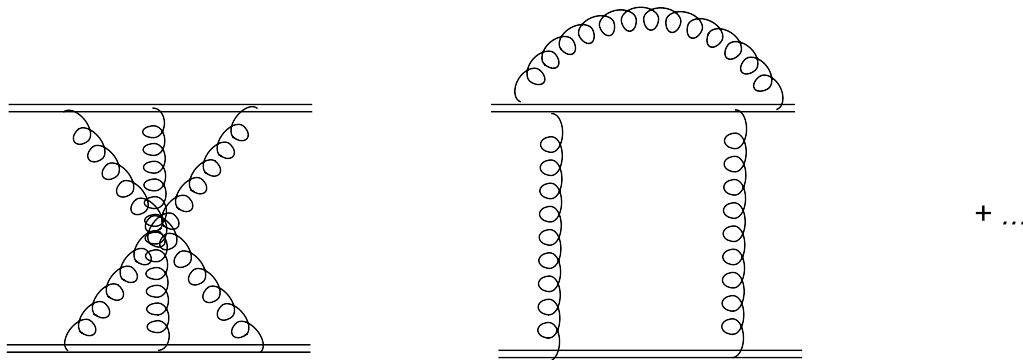
Tree and one-loop approximations:



$$F(a_1, a_2, a_3) = \int \frac{d^d k}{(-k^2 - i0)^{a_1} (-(q - k)^2 - i0)^{a_2} (-v \cdot k - i0)^{a_3}}$$

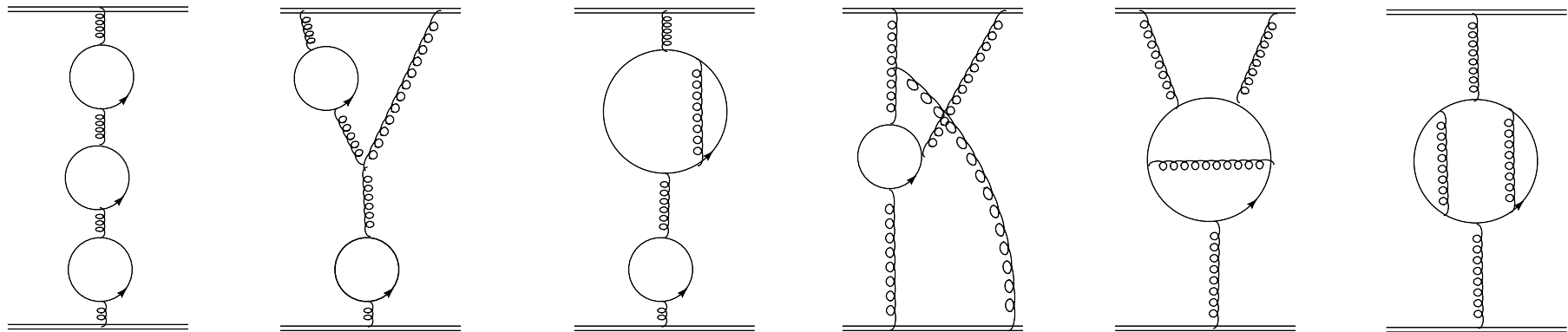
with $q \cdot v = 0$, $v = (1, \vec{0})$

Two loops:



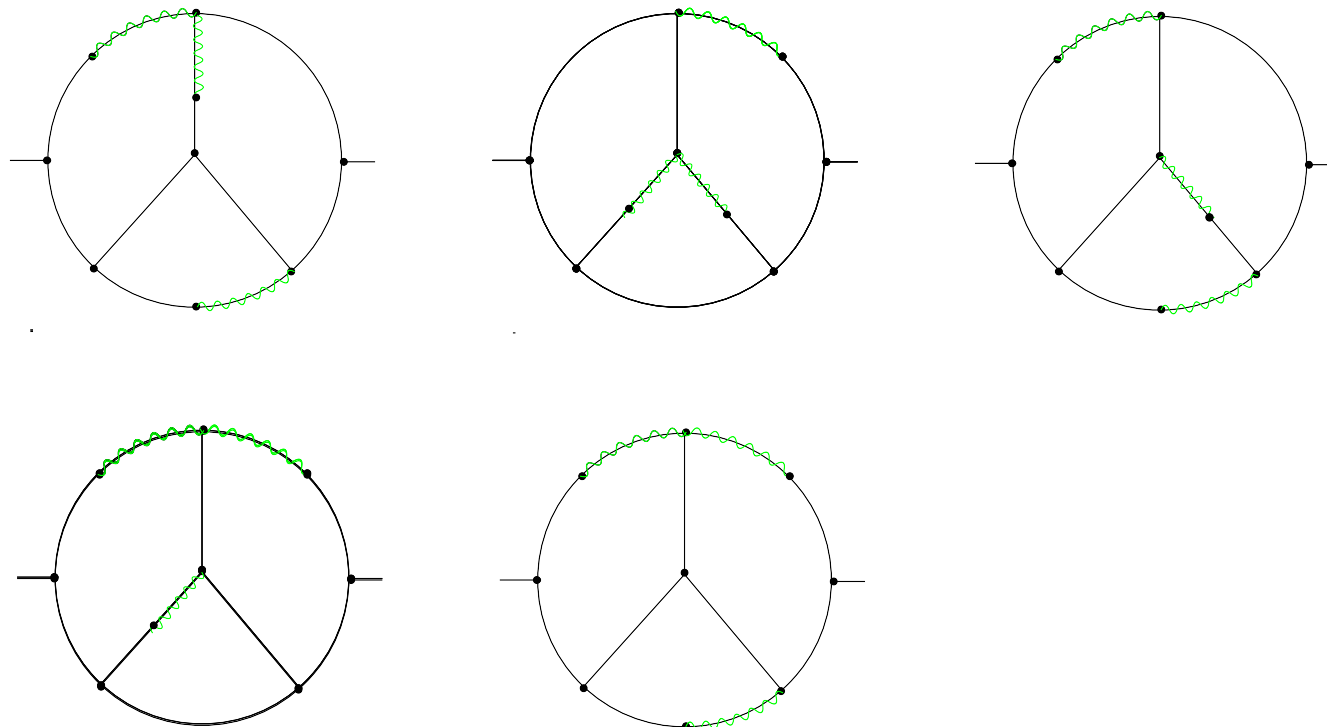
Generation of diagrams by QGRAPH

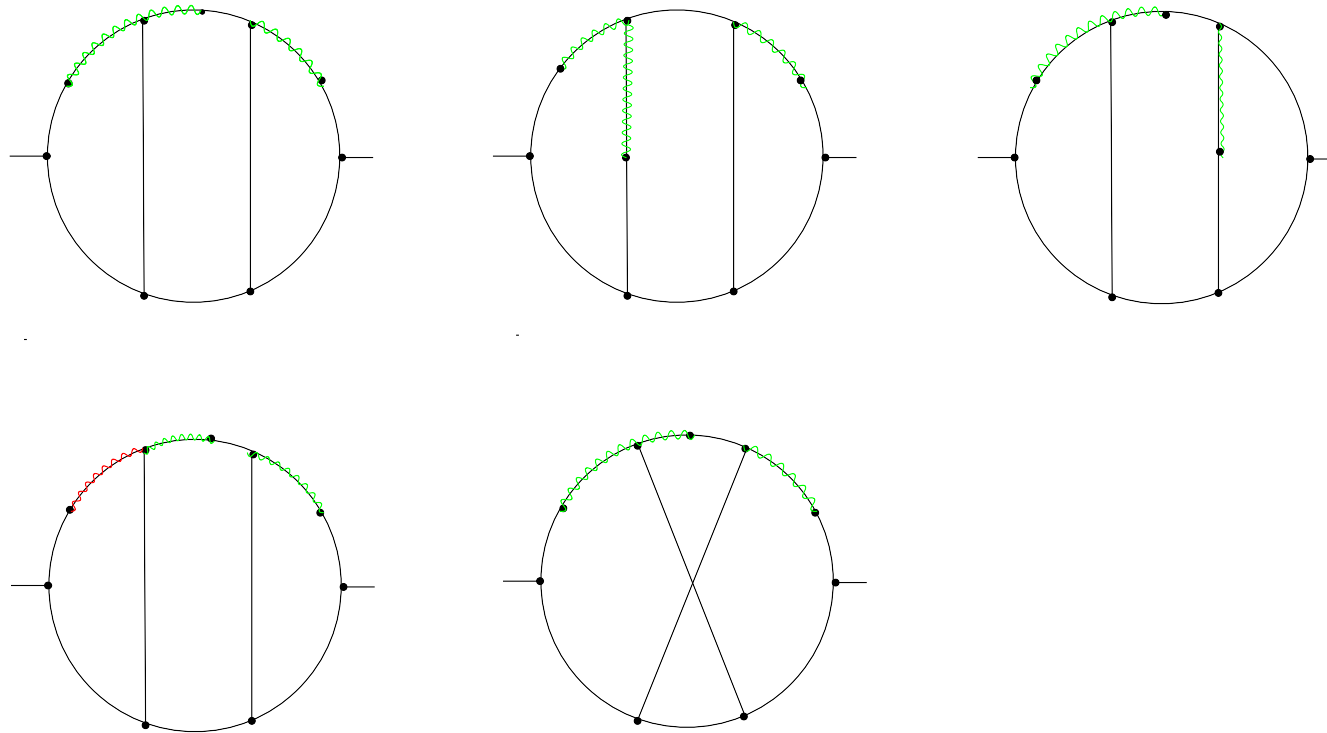
Typical diagrams in three loops:


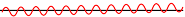
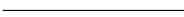


$$C_F(n_l T_F)^3 \quad C_F C_A (n_l T_F)^2 \quad C_F^2 (n_l T_F)^2 \quad C_F C_A^2 n_l T_F \quad C_F^2 C_A n_l T_F \quad C_F^3 n_l T_F$$

Various classes of Feynman integrals with 12 indices:



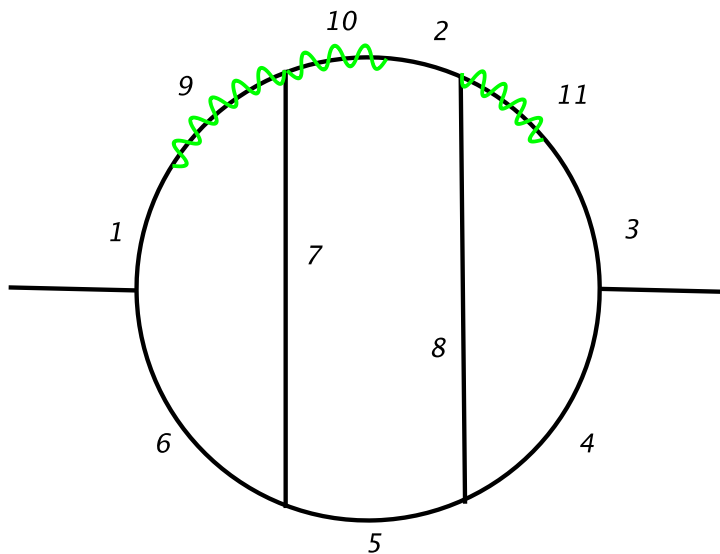


		
$\frac{1}{(-v \cdot k - i0)^a}$	$\frac{1}{(-v \cdot k + i0)^a}$	$\frac{1}{(-k^2 - i0)^a}$

And 15 types of two-loop Feynman integrals with 7 indices, where one index can be shifted: $a_i \rightarrow a_i + \epsilon$ or $a_i \rightarrow a_i + 2\epsilon$

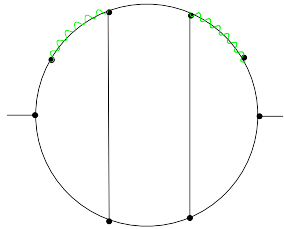
For n_l^1 contribution: ~ 70000 integrals of different types in the general ξ -gauge

For example,

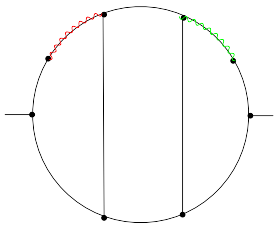


with the numerator chosen as $(-(k - r)^2)^{-a_{12}}$

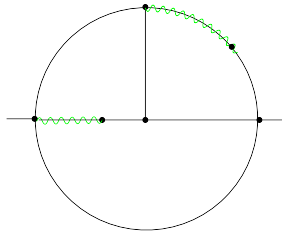
Evaluating master integrals by MB.
 For example,



$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2} \left[\frac{56\pi^4}{135\epsilon} + \frac{112\pi^4}{135} + \frac{16\pi^2\zeta(3)}{9} + \frac{8\zeta(5)}{3} + O(\epsilon) \right]$$



$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2} \left[-\frac{64\pi^4}{135\epsilon} - \frac{128\pi^4}{135} - \frac{32\pi^2\zeta(3)}{9} + \frac{8\zeta(5)}{3} + O(\epsilon) \right]$$



$$\frac{(i\pi^{d/2})^3}{q^2 v^2} \left[\frac{32\pi^4}{135\epsilon} - \frac{128\pi^4}{135} + \frac{88\pi^2\zeta(3)}{9} + \frac{188\zeta(5)}{3} + O(\epsilon) \right]$$

Transcendentality level 5 (and, sometimes, 6) is needed.
 The constants of level 5 that we encounter:

$$\zeta(5), \zeta(4) \ln 2, \zeta(2)\zeta(3), \zeta(3) \ln^2 2, \zeta(2) \ln^3 2, \\ \ln^5 2, \text{Li}_5\left(\frac{1}{2}\right), \text{Li}_4\left(\frac{1}{2}\right) \ln 2$$

$$a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)},$$

[A.V. Smirnov, V.A. Smirnov, and M. Steinhauser'08]

$$a_3^{(3)} = - \left(\frac{20}{9} \right)^3 T_F^3,$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135} \right) C_A T_F^2,$$

$$+ \left(\frac{14002}{81} - \frac{416\zeta(3)}{3} \right) C_F T_F^2,$$

$$\begin{aligned}
a_3^{(1)} &= (-709.717) C_A^2 T_F + \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5) \right) C_A C_F T_F \\
&+ \left(\frac{286}{9} + \frac{296\zeta(3)}{3} - 160\zeta(5) \right) C_F^2 T_F + (-56.83(1)) \frac{d_F^{abcd} d_F^{abcd}}{N_A},
\end{aligned}$$

$$C_A = N_c, \quad C_F = \frac{N_c^2 - 1}{2N_c}, \quad T_F = \frac{1}{2},$$

$$\frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{18 - 6N_c^2 + N_c^4}{96N_c^2}.$$

$a_3^{(0)}$ to be done.