Perturbative Results Without Diagrams

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1. Introduction

Usually in perturbative calculation in Quantum (Field) Theory the number of diagrams grows factorially with the order

Example: number of diagrams for g-2 of the electron in QED

(see Itzykson & Zuber p. 466, 467)

$$\Gamma(\alpha) = \frac{4z(1-S)}{S^3}\,, \quad S = -2z\left[1+\frac{K_0'(z)}{K_0(z)}\right]\,, \quad z = -\frac{1}{4\alpha}$$

expand in powers of $\alpha = 1/137.036$

$$\Longrightarrow \quad \Gamma(\alpha) = 1 + \alpha + 7\,\alpha^2 + 72\,\alpha^3 + 891\,\alpha^4 + 12672\,\alpha^5 + 202770\,\alpha^6 + \ldots$$

Consequence: huge cancellations between individual diagrams

heroic efforts needed for higher-order calculations

Schwinger (1948), Petermann, Sommerfield (1957)

Laporta & Remiddi (1996), Kinoshita et al. (1990-2005)

Need (more modestly: would be nice to have) new methods !

2. A new method (applied to the polaron g. s. energy)

Take as simple (but nontrivial) example the **polaron** problem – a non-relativistic field theory

polaron = electron slowly moving through polarizable crystal

model Hamiltonian H. Fröhlich (1954)

$$\hat{H} \sim \frac{1}{2}\hat{\mathbf{p}}^2 + \sum_k \hat{a}_k^\dagger \hat{a}_k + \sqrt{\alpha} \sum_k \frac{1}{|\mathbf{k}|} \left[\hat{a}_k^\dagger e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} + h.c. \right]$$

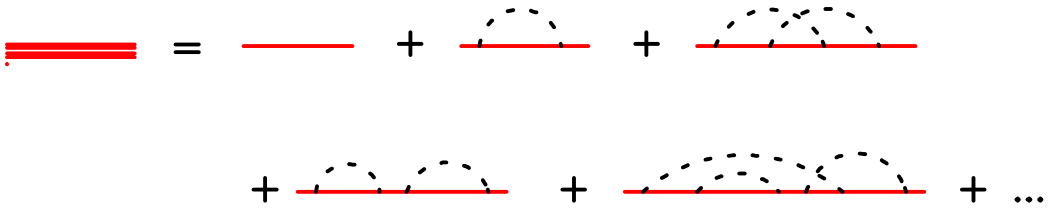
α: dimensionless electron-phonon coupling constant

Ground-state energy of polaron:

$$E_0 := \sum_{n=1} \mathbf{e_n}\, \alpha^n \quad , \quad \mathbf{e_1} = -1$$

$$\mathbf{e_2} = -0.01591962 \text{ (1959)} , \quad \mathbf{e_3} = -0.00080607 \text{ Smondyrev (1986)}$$

In field-theoretic language: have to evaluate self-energy diagrams with more and more loops



Long live the PATH INTEGRAL : phonons can be integrated out exactly!
Feynman (1955)

$$Z(\beta) \;=\; \oint \mathcal{D}^3 x \, e^{-S_{\rm eff}} \;\xrightarrow{\beta\to\infty}\; e^{-\beta E_0}$$

where for large β

$$\Rightarrow \; S_{\rm eff}[x] \;\sim\; \int_0^\beta dt \, \frac{1}{2}\dot{\mathbf{x}}^2 + \alpha \int_0^\beta dt_1 dt_2 \, e^{-|t_1-t_2|} \int d^3k \, \frac{1}{\mathbf{k}^2} \exp\left[i\mathbf{k}\cdot(\mathbf{x}(t_1)-\mathbf{x}(t_2))\right] \;=: \; S_0 + S_1$$

Employ **cumulant expansion** of partition function

$$Z(\beta) \; = \; Z_0 \exp \left[\sum_{n=1} \frac{(-)^n}{n!} \lambda_n(\beta) \right]$$

where $\lambda_n(\beta)$ are the cumulants w.r.t. S_1

Recursion relation with the moments $m_n(\beta) \; \equiv \; \langle \, S_1{}^n \, \rangle \; \propto \; \alpha^n$

$$\lambda_{n+1} \; = \; m_{n+1} - \sum_{k=0}^{n-1} \binom{n}{k} \lambda_{k+1} \, m_{n-k}$$

$$\begin{aligned}
\lambda_1 &= m_1 \\
\lambda_2 &= m_2 - m_1^2 \\
\lambda_3 &= m_3 - 3\, m_2\, m_1 + 2\, m_1^3 \\
\lambda_4 &= m_4 - 4\, m_3\, m_1 - 3\, m_2^2 + 12\, m_2\, m_1^2 - 6\, m_1^4 \\
\lambda_5 &= m_5 - 5\, m_4\, m_1 - 10\, m_3\, m_2 + 20\, m_3\, m_1^2 + 30\, m_2^2\, m_1 - 60\, m_2\, m_1^3 + 24\, m_1^5 \\
&\vdots
\end{aligned}$$

$$\textbf{Note}: \; \lambda_n(\beta) \; \propto \; \alpha^n \qquad \Longrightarrow \qquad \mathbf{e_n} \; = \; \lim_{\beta \to \infty} \frac{1}{\beta} \frac{(-)^{n+1}}{\alpha^n \, n!} \lambda_n(\beta)$$

The path integral for the moments

$$m_n \;=\; C \oint \mathcal{D}^3x \, S_1{}^n \, e^{-S_0[x]} \quad , \quad m_0 \;=\; 1$$

can be evaluated exactly. Write **Coulomb propagator** as

$$\frac{1}{\mathbf{k}^2} = \frac{1}{2}\int_0^\infty du \, \exp\left[-\frac{1}{2}\mathbf{k}^2 \, u\right]$$

$\Longrightarrow$ **all** momentum integrations can be performed and one obtains

$$m_n \;=\; (-)^n \frac{\alpha^n}{(4\pi)^{n/2}} \prod_{m=1}^n \left(\int_0^\beta dt_m \int_0^{t_m} dt'_m \int_0^\infty du_m \right) \exp\left[- \sum_{m=1}^n (t_m - t'_m) \right]$$
$$\cdot \left[\det \mathbf{A} \left(t_1 \ldots t_n, t'_1 \ldots t'_n; u_1 \ldots u_n \right) \right]^{-3/2}$$

with $(n \times n)$- matrix $\mathbf{A}$

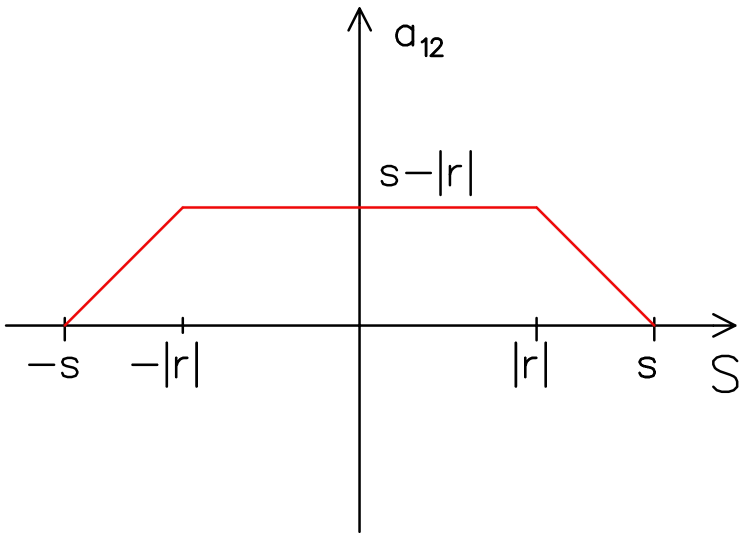
$$\mathbf{A}_{ij} \;=\; \frac{1}{2} \Big[\underbrace{-|t_i - t_j| + |t_i - t'_j| + |t'_i - t_j| - |t'_i - t'_j|}_{\text{non-analytic}} \Big] + \underbrace{u_i \, \delta_{ij}}_{\text{analytic dependence}}$$

Define

$$\mathbf{A}_{ij} \; =: \; \mathbf{a}_{ij} + u_i \, \delta_{ij}$$

Diagonal parts: $\mathbf{a}_{ii} \; = \; t_i - t'_i \; \equiv \; \sigma_i$

Non-diagonal matrix elements :



$$\begin{aligned} S \; &:= \; \frac{1}{2} \Big(t_i + t'_i - (t_j + t'_j) \Big) \\ r \; &:= \; \frac{1}{2} \left(\sigma_i - \sigma_j \right) \\ s \; &:= \; \frac{1}{2} \left(\sigma_i + \sigma_j \right) \end{aligned}$$

3. Numerical procedures and results

For $m_n(\beta) \Rightarrow \lambda_n(\beta)$ one has to do a $\mathbf{3n}$-dimensional integral over t_i, t'_i, u_i

Two u_i-integrations can be done analytically

$$\int_0^\infty du_n \, \mathrm{det}_n^{-3/2} A(1,2,\ldots,n) \;=\; \frac{2}{A_n \sqrt{\mathrm{det}_n \, A(u_n = 0)}}$$

$$\int_0^\infty du_{n-1} \int_0^\infty du_n \, \mathrm{det}_n^{-3/2} A(1,2,\ldots,n) \;=\; \frac{4}{\sqrt{A_{n-1,n} \, A_{n-1} \, A_n}} \, \frac{\arcsin \sqrt{x_{HF}}}{\sqrt{x_{HF}}}$$

where $A_n, A_{n-1}, A_{n-1,n}$ are **principal minors** of the determinant $\mathrm{det}_n \, A \equiv A$

$$0 \;\le\; x_{HF} \;:=\; 1 - \frac{A_{n-1,n} \, A}{A_{n-1} \, A_n} \;\le\; 1$$

because A_{ij} is a **positive semi-definite** matrix $\Longrightarrow$ Hadamard-Fischer inequality

$$A_{n-1} \, A_n \;\ge\; A_{n-1,n} \, A$$

After performing u_n, u_{n-1}-integrations analytically

$\Longrightarrow$ $(\mathbf{3n - 2})$-dimensional integral left

Useful trick: calculate directly $\frac{\partial \lambda_n}{\partial \beta} \Longrightarrow (\mathbf{3n-3})$-dimensional integral !

Further advantage: asymptotic behaviour (thus extrapolation to $\beta \to \infty$) is much improved:

$$\mathbf{e_n}(\beta) \; := \frac{(-)^{n+1}}{\alpha^n \, n!} \frac{\partial \lambda_n}{\partial \beta} = \frac{\partial}{\partial \beta} \left[\beta \cdot \mathbf{e_n} + \cancel{\text{const}} - \ldots \right] \stackrel{\beta \to \infty}{\longrightarrow} \mathbf{e_n} - \frac{a_n}{\sqrt{\beta}} \, e^{-\beta} \, (?) + \ldots$$

Exponential convergence to $\mathbf{e_n}$: analytically proved for $n = 1, 2$

numerically for $n = 3$:

assume $\mathbf{e_n}(\beta) \to \mathbf{e_n} - a_n \, \beta^{-\kappa_n} \, e^{-\beta}$

fit to Monte-Carlo data gives $\kappa_3 = 0.55(3)$

Assume it also for $n > 3$ ...

Numerical evaluation:

mapping to hypercube $[0, 1]$, then:

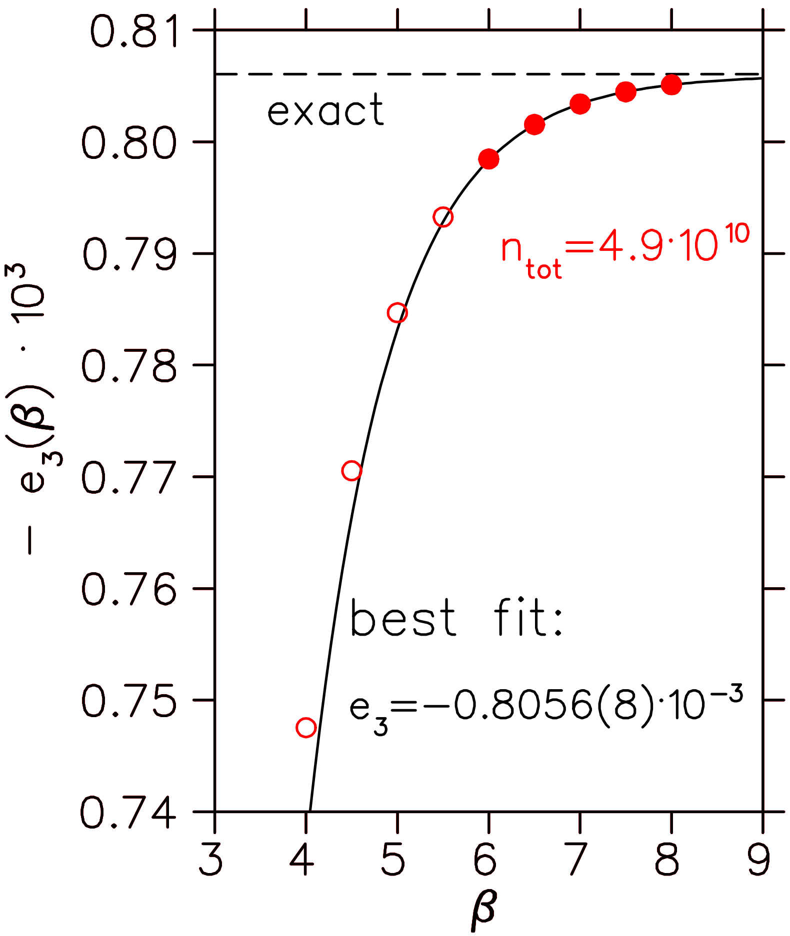
Monte-Carlo integration

with **VEGAS** program or routines from the **CUBA** library

Note: Monte-Carlo integration can handle non-analytic,

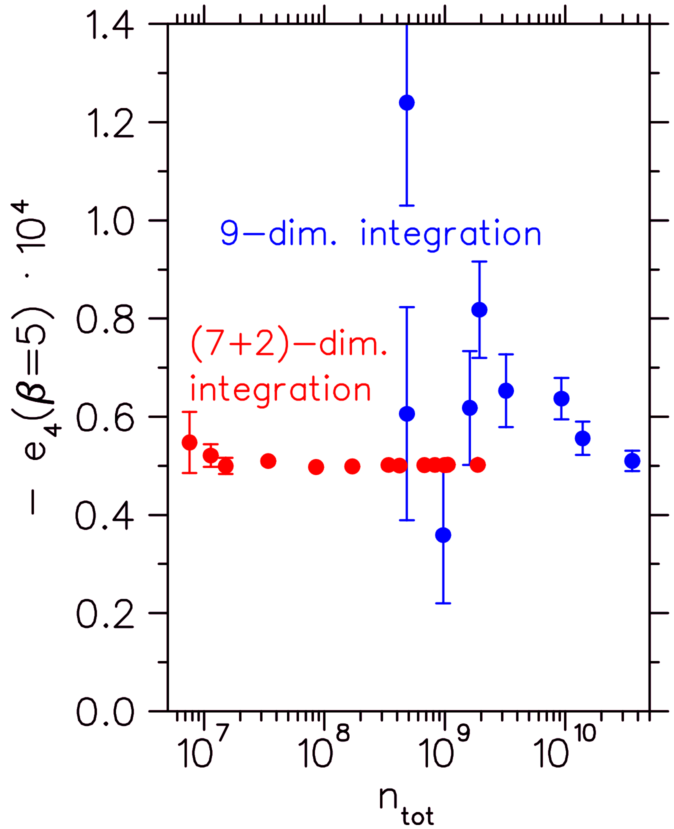
even discontinous integrands

check $n = 3$ (6-dimensional integral):

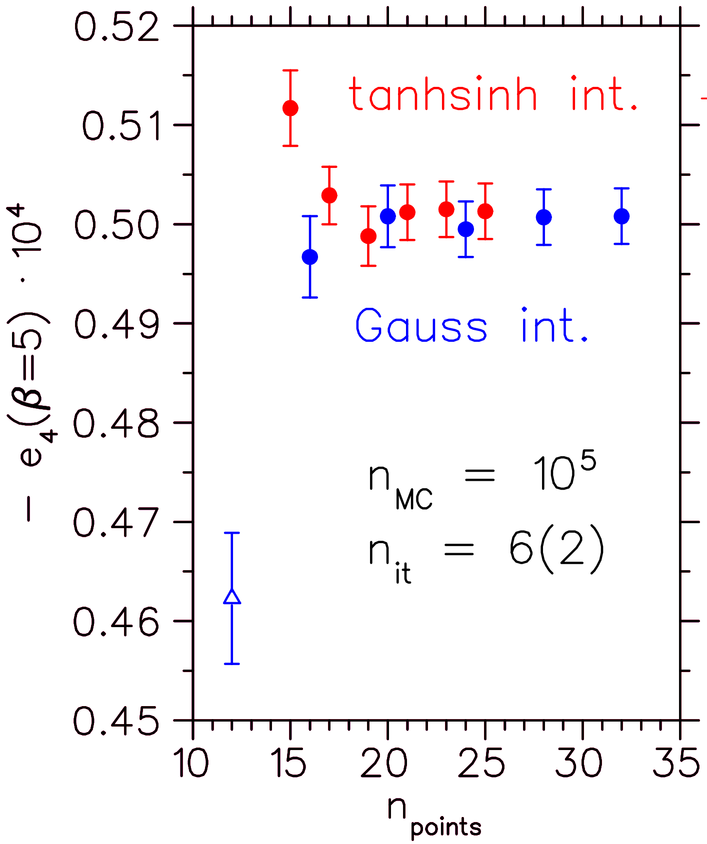


analytical: $e_3 = -0.80607005 \cdot 10^{-3}$

but for $n = 4$ convergence is slow with number of function calls:



solution: perform the $(n-2)$ remaining u_i -integrations by deterministic integration routine. Very efficient: **tanh-sinh-method** !



$\longrightarrow$ Transformation

$$x = g(t) = \tanh(\sinh t) \qquad t \in [-\infty, +\infty]$$

$$g'(t) = \frac{\cosh t}{\cosh^2(\sinh t)}$$

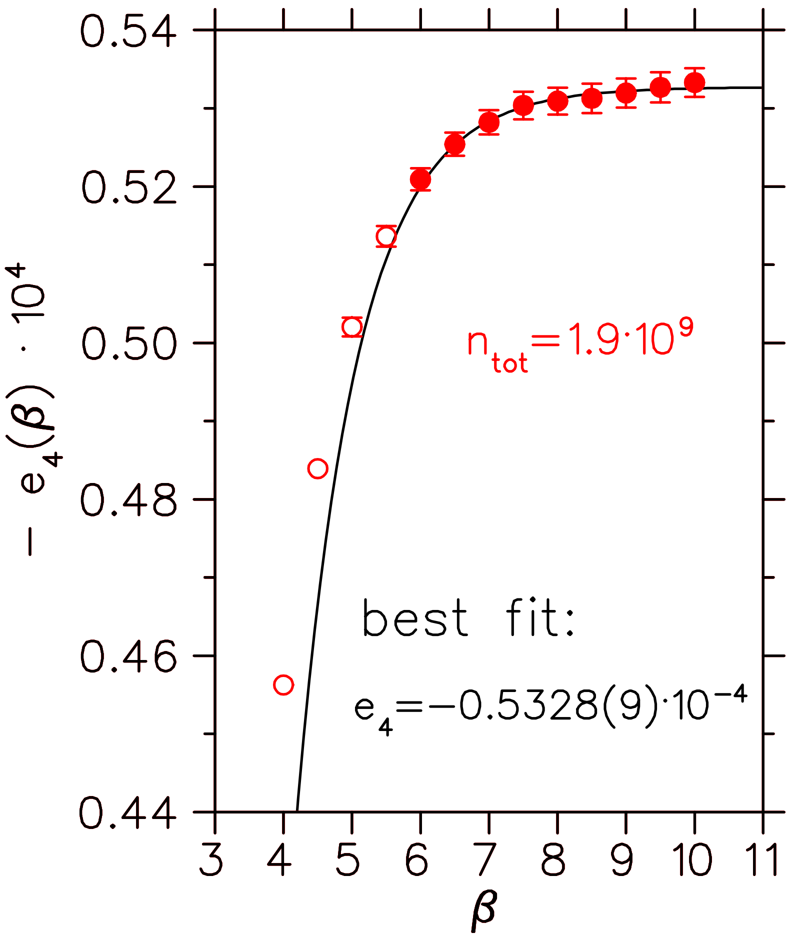
Euler – MacLaurin $\Longrightarrow$

$$\int_{-1}^{+1} dx\, f(x) = \int_{-\infty}^{+\infty} dt\, g'(t)\, f(g(t)) \approx h \sum_{k=-\infty}^{k=+\infty} w_k\, f(x_k)$$

with $x_k = g(kh)$, $w_k = g'(kh)$

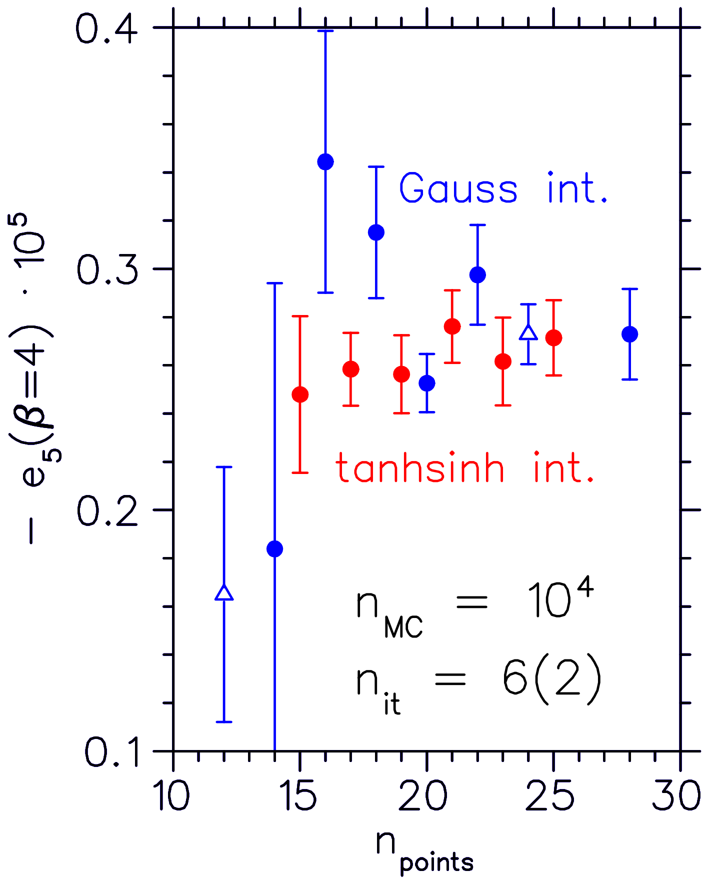
w_k double exponentially decreasing for large $|k|$

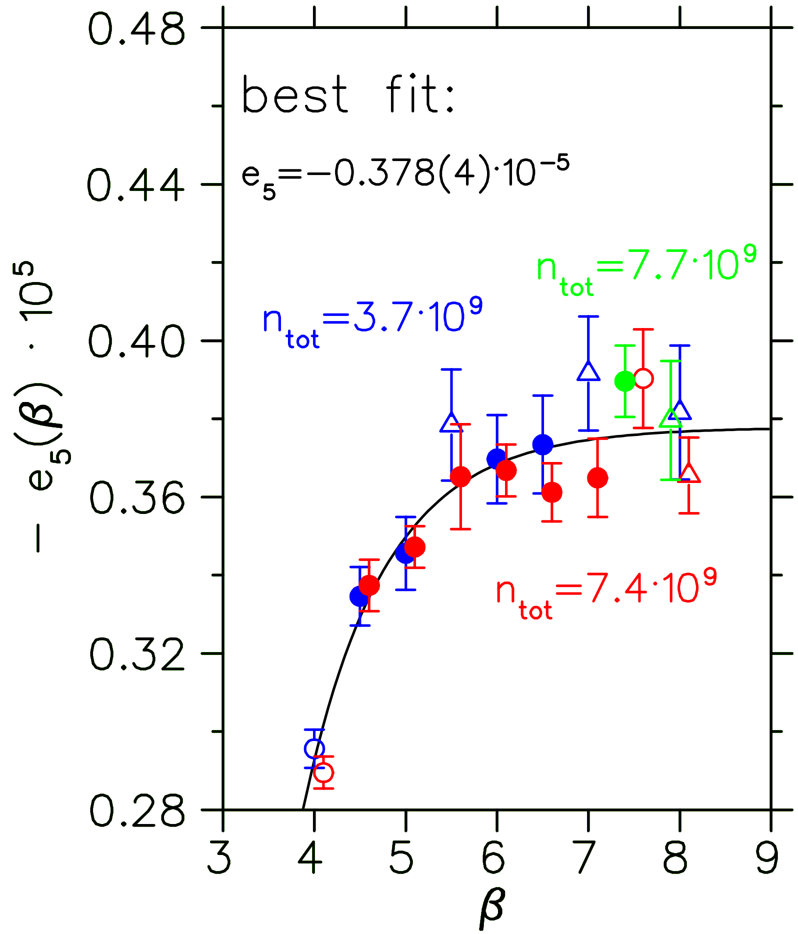
now for $n = 4$:


0.54
0.52
0.50
0.48
0.46
0.44
3 4 5 6 7 8 9 10 11
β
− e4(β) · 10^4
ntot=1.9·10^9
best fit:
e4=−0.5328(9)·10^−4

and also $n = 5$ (evaluation of (9+3)-dim. integral) is within reach:

Here tanh-sinh-integration seems to be slightly better than Gauss-Legendre




best fit:
$e_5 = -0.378(4) \cdot 10^{-5}$
$n_{tot} = 3.7 \cdot 10^9$
$n_{tot} = 7.7 \cdot 10^9$
$n_{tot} = 7.4 \cdot 10^9$
$-e_5(\beta) \cdot 10^5$
β
0.48
0.44
0.40
0.36
0.32
0.28
3
4
5
6
7
8
9

4. Summary

- Two additional perturbative coefficients $\mathbf{e_4}$, $\mathbf{e_5}$ for the polaron g.s. energy have been determined by a new (mostly) numerical method. This amounts to performing a **4-loop** and **5-loop** calculation in Quantum Field Theory

- Method is based on a combination of **Monte-Carlo integration** techniques and deterministic **quadrature rules** for finite β (temperature) and on judicious extrapolation to $\beta \to \infty$ (zero temperature). Reproduces Smondyrev's coefficient $\mathbf{e_3}$ with high accuracy

- Cancellation in n$^{\text{th}}$ order **not** among many individual diagrams but among much fewer terms in the integrand of the $(3n-3)$-dimensional integral

- Increased **computational power** would allow to improve accuracy for $\mathbf{e_4}$, $\mathbf{e_5}$ and even $\mathbf{e_6}$ seems accessible

- Application to **g-2 of the electron** under investigation (worldline representation of QED). Challenge: renormalization !?

5. Outlook: application to worldline QED

Generating functional

$$
\begin{aligned}
Z[\bar{\eta},\eta,j] &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \ \exp\Big\{ i\, S[\bar{\psi},\psi,A] + (\bar{\psi},\eta) + (\bar{\eta},\psi) + (j,A) \Big\} \\
S[\bar{\psi},\psi,A] &= \left(\bar{\psi},\, \Big[\gamma \cdot (\underbrace{i\partial - eA}_{\equiv \Pi}) - M_0\Big]\, \psi \right) + S_0[A]
\end{aligned}
$$

2-point function:

$$
\big\langle\, \psi(x)\bar{\psi}(0)\,\big\rangle \sim \int \mathcal{D}A \left\langle x \left| \underbrace{\frac{1}{\gamma \cdot \Pi - M_0}} \right| 0 \right\rangle e^{iS_0[A]} \ \underbrace{\text{Det}\,(\gamma \cdot \Pi - M_0)}
$$

$\downarrow$ **Schwinger trick** $\qquad\qquad$ $\uparrow$ $=$ const. in **quenched approx.**

$\downarrow$

$$
\sim \int_0^\infty dT \int d\chi \ \exp\Big[i \left((\gamma \cdot \Pi)^2 - M_0^2 \right) T - i\,(\gamma \cdot \Pi + M_0)\,\chi \Big] \qquad \int d\chi = 0\,, \int d\chi\,\chi = 1
$$

Berezin

result in momentum space : (Alexandrou, RR & Schreiber, PR **A** 59 (1999))

$$G_2(p) \sim \int_0^\infty dT \int d\chi \, e^{-i(M_0^2 T + M_0 \chi)} \int d^4x \, e^{-ip\cdot x} \int \mathcal{D}A \, e^{iS_0[A]} \int \mathcal{D}^4x \int \mathcal{D}^4\zeta \;\; e^{i\,S[x,\zeta,\chi,A]} \Big|_{\Gamma\to\gamma}$$

orbital trajectory : $x(0) = 0\,,\ x(T) = x$

spin trajectory : $\zeta(0) + \zeta(T) = \Gamma$

$$S[x,\zeta,\chi,A] \quad \sim \quad \int_0^T dt \left(-\frac{1}{2}\dot{x}^2 + i\zeta\cdot\dot{\zeta} + \dot{x}\cdot\zeta\,\chi - e\,\dot{x}\cdot A - ie\,\zeta\cdot F\cdot\zeta \right)$$

↑ spin-orbit ↑ convection ↑ spin current

Photon field A can be integrated out exactly $\Longrightarrow$ effective action

In supersymmetric formulation: $t \to t, \theta$

$$X_\mu \quad :\sim \quad x_\mu + \theta\, \zeta_\mu \quad , \qquad D := \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t}$$

$$S_{\text{eff}}[x, \zeta, \chi] \sim S_0 + \alpha \int dt_1 d\theta_1\, dt_2 d\theta_2 \int d^4k\, \frac{1}{k^2}\, DX_1 \cdot DX_2 \exp\left[-ik \cdot (X_1 - X_2)\right]$$

———————————————— ↖ free photon propagator

2-point function

$$G_2(p) \sim \int_0^\infty d\mathbf{T} \int d\chi \exp\left[i\, A(p, \Gamma, \mathbf{T}) + iB(p, \Gamma, \mathbf{T})\, \chi \right]$$

develops pole at $\gamma \cdot p = M_{\text{phys}}$ if for large $\mathbf{T}$

$$\begin{aligned} A(p, \Gamma, \mathbf{T}) \quad &\longrightarrow \quad \left(p^2 - M_{\text{phys}}^2 \right) \mathbf{T} \\ B(p, \Gamma, \mathbf{T}) \quad &\longrightarrow \quad Z_2 \left(p \cdot \Gamma + M_{\text{phys}} \right) \end{aligned}$$

————————————————

Determine M_{phys} and wavefunction renorm. Z_2 – similar to polaron case !

From 3-point function $< \psi \bar{\psi} A >$: electron-photon coupling, i.e. **anomalous magnetic moment of electron**