Determining the global minimum of extended Higgs potentials

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in collaboration with

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Villigen, 10 December 2007

Motivation

Why extended Higgs sectors ?

- supersymmetry (naturalness, coupl. unification, “highest symmetry”)
- baryogenesis
- why not ?

Why bother with global minimisation ?

- vacuum should be global minimum / not rapidly decaying
- high-dimensional non-compact field space:
  local min. / partial checks no guarantee

Outline

1. General two-Higgs-doublet model
   - THDM potential via gauge invariant functions
   - Minima
   - CP symmetries
2. NMSSM
   - NMSSM Potential
   - Gröbner bases
   - Minima

- Standard Model (SM) contains one complex Higgs doublet $\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}$

- Two-Higgs-Doublet Model (THDM) as “simplest ext.”: $\varphi_1 = \begin{pmatrix} \varphi_1^+ \\ \varphi_1^0 \end{pmatrix}, \varphi_2 = \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix}$

- new features:
  - physical Higgses: 3 neutral, 1 charged pair
  - explicite / spontaneous CP violation

- literature on THDMs: huge amount; recently (still incomplete):
  - basis indep.: Gunion, Haber, Davidson; Nishi; Feirreira, Santos, Barroso; Branco, Rebelo, Silva-Marcos; Ivanov; Ginzburg, Krawczyk
  - CP viol.: Botella, Lavoura, Silva; Khater, Osland
  - specific models: Barbieri, Hall

- here: method for analysis of *most general* THDM
  - basis indep. structural statements (incl. “new CP symm.”)
  - access to minima

THDM Higgs Potential

How can we describe the most general THDM ?

- two complex Higgs-doublet fields with hypercharge $y = +1/2$:

$$\varphi_1(x) = \begin{pmatrix} \varphi_1^+(x) \\ \varphi_1^0(x) \end{pmatrix}, \qquad \varphi_2(x) = \begin{pmatrix} \varphi_2^+(x) \\ \varphi_2^0(x) \end{pmatrix}$$

- renormalisable, gauge invariant potential contains only

$$\varphi_i^\dagger \varphi_j, \quad \left(\varphi_i^\dagger \varphi_j\right)\left(\varphi_k^\dagger \varphi_l\right), \qquad i, j, k, l \in \{1, 2\}$$

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$$\varphi_i^\dagger \varphi_j, \quad (\varphi_i^\dagger \varphi_j)(\varphi_k^\dagger \varphi_l), \qquad i,j,k,l \in \{1,2\}$$

Definition

real gauge invariant functions K_0, K_1, K_2, K_3:

$$\begin{pmatrix} \varphi_1^\dagger \varphi_1 & \varphi_2^\dagger \varphi_1 \\ \varphi_1^\dagger \varphi_2 & \varphi_2^\dagger \varphi_2 \end{pmatrix} \equiv \frac{1}{2} \left(K_0 \, \mathbb{1} + K_a \, \sigma^a \right) \quad \Leftrightarrow \quad \begin{cases} K_0 = \varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2, & K_1 = 2 \, \mathrm{Re} \, \varphi_1^\dagger \varphi_2, \\ K_3 = \varphi_1^\dagger \varphi_1 - \varphi_2^\dagger \varphi_2, & K_2 = 2 \, \mathrm{Im} \, \varphi_1^\dagger \varphi_2 \end{cases}$$

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- general THDM Higgs potential for $\tilde{\boldsymbol{K}}^{\mathrm{T}} \equiv (K_0, K_1, K_2, K_3) \equiv (K_0, \boldsymbol{K})$:

$$V = \tilde{\xi}^{\mathrm{T}} \tilde{\boldsymbol{K}} + \tilde{\boldsymbol{K}}^{\mathrm{T}} \tilde{E} \tilde{\boldsymbol{K}}$$

with $\tilde{\xi}$, $\tilde{E} = \tilde{E}^{\mathrm{T}}$ real parameters

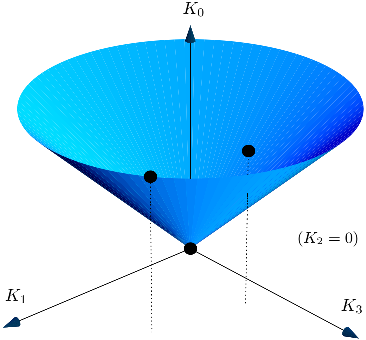
- no gauge d.o.f. in this scheme, reduced powers

Gauge invariant functions

$$\begin{pmatrix} \varphi_1^\dagger\varphi_1 & \varphi_2^\dagger\varphi_1 \\ \varphi_1^\dagger\varphi_2 & \varphi_2^\dagger\varphi_2 \end{pmatrix} \equiv \frac{1}{2}\left(K_0\,\mathbb{1} + K_a\,\sigma^a\right) \quad \Leftrightarrow \quad \begin{cases} K_0 = \varphi_1^\dagger\varphi_1 + \varphi_2^\dagger\varphi_2, & K_1 = 2\,\mathrm{Re}\,\varphi_1^\dagger\varphi_2, \\ K_3 = \varphi_1^\dagger\varphi_1 - \varphi_2^\dagger\varphi_2, & K_2 = 2\,\mathrm{Im}\,\varphi_1^\dagger\varphi_2 \end{cases}$$

- domain:

$$K_0 \geq 0$$
$$K_0^2 - \boldsymbol{K}^2 \geq 0$$



Minkowski type structure: $(K_0, \boldsymbol{K})$
on & inside "forward light cone"

Gauge invariant functions

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- domain:

$$K_0 \geq 0$$
$$K_0^2 - \boldsymbol{K}^2 \geq 0$$

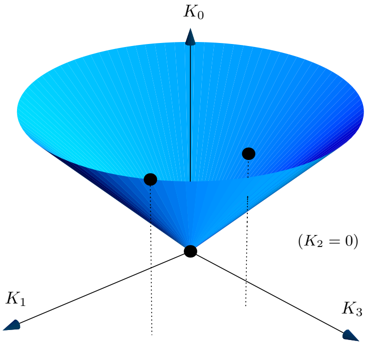
- change of doublet basis by $U \in U(2)$

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = U \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

means for gauge invariant functions

$$K_0' = K_0,$$
$$\boldsymbol{K}' = R(U)\,\boldsymbol{K},$$

with $R(U) \in SO(3)$



Minkowski type structure: $(K_0, \boldsymbol{K})$ on & inside "forward light cone"

Stationary points
Classes

- assume V bounded from below (see later), then global minimum amongst stat. pnts.
- potential $V = \tilde{\xi}^{\mathrm{T}} \tilde{\boldsymbol{K}} + \tilde{\boldsymbol{K}}^{\mathrm{T}} \tilde{E} \tilde{\boldsymbol{K}}$
- three classes of stationary points (i.e. of minima, maxima, saddles):

$K_0 = \boldsymbol{K} = 0$:

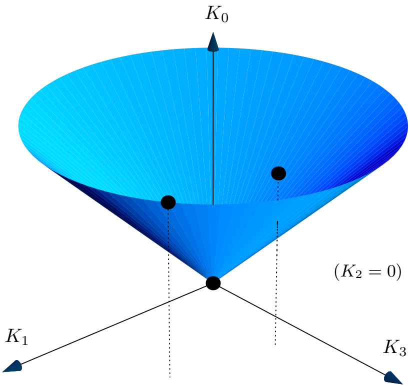
trivial solution ($\varphi_1 = \varphi_2 = 0$)

$K_0 > \boldsymbol{K}$:

solve $\nabla_{\tilde{\boldsymbol{K}}} V = 0$

$K_0 = \boldsymbol{K} > 0$:

solve $\nabla_{\tilde{\boldsymbol{K}},u} \left[V - u(K_0^2 - \boldsymbol{K}^2) \right] = 0$



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$K_0 = \boldsymbol{K} = 0$:

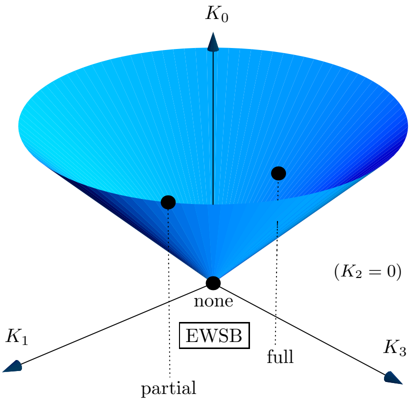
trivial solution ($\varphi_1 = \varphi_2 = 0$)
unbroken $SU(2)_L \times U(1)_Y$

$K_0 > \boldsymbol{K}$:

solve $\nabla_{\tilde{\boldsymbol{K}}} V = 0$
fully broken $SU(2)_L \times U(1)_Y$

$K_0 = \boldsymbol{K} > 0$:

solve $\nabla_{\tilde{\boldsymbol{K}},u} \left[V - u(K_0^2 - \boldsymbol{K}^2) \right] = 0$
$SU(2)_L \times U(1)_Y$ broken to $U(1)_{em}$



Stationary points
Solutions

combined stat. conditions for non-trivial classes (set $u = 0$ for full breaking):

$$\nabla_{\tilde{\boldsymbol{K}}}(V - u\,\tilde{\boldsymbol{K}}^{\mathrm{T}}\tilde{g}\tilde{\boldsymbol{K}}) = \tilde{\xi} + 2(\tilde{E} - u\,\tilde{g})\tilde{\boldsymbol{K}} = 0, \qquad u\,\tilde{\boldsymbol{K}}^{\mathrm{T}}\tilde{g}\tilde{\boldsymbol{K}} = 0,$$

where $\tilde{g} = \mathrm{diag}(1, -1, -1, -1)$

- solutions: linear algebra + zeros of 1 univariate polynomial
  (potential vals + generic stat. points via one univ. function)
- general reparametrisation possible
- applicable in any basis

Minima

general expressions easily give:

- mutually exclusive possibilities for local minima:
  - one or multiple min. with required EWSB ($K_0 = |\boldsymbol{K}|$)
  - one charge breaking minimum ($K_0 > |\boldsymbol{K}|$)
  - (degenerate set of solutions ($K_0 \geq |\boldsymbol{K}|$))
  - trivial minimum ($\tilde{\boldsymbol{K}} = 0$)

  (field-based hierachy consid.: Ferreira, Santos, Barroso; Ivanov)

- nontrivial minimum $\Leftrightarrow \xi_0 < |\boldsymbol{\xi}|$,

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Theorem

Global minimum with spont. symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{em}$

- *is given and guaranteed by stat. pnt. of type $K_0 = |\boldsymbol{K}| > 0$ with largest Lagrange multiplier $u_0 > 0$. It holds then*

$$m_{H^\pm}^2 = 2u_0 v_0^2$$

Generalised CP transformations

- consider generalised CP transformations

$$\mathrm{CP}_g: \quad \varphi_i \to U_{ij}\,\varphi_j^* \qquad \Rightarrow \qquad \begin{aligned} &K_0 \to K_0 \\ &\boldsymbol{K} \to \bar{R}_U\,\boldsymbol{K}, \qquad \bar{R}_U \in O(3)\,, \quad \det \bar{R}_U = -1 \end{aligned}$$

(see also: Ecker, Grimus, Neufeld 1987)

- require $(\mathrm{CP})^2$ to be symmetry of Lagrangian

$$\bar{R}_U\,\bar{R}_U = \mathbb{1}$$

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$$\bar{R}_U\,\bar{R}_U = \mathbb{1}$$

- two types of generalised CP transformations
  - (i) point reflection at origin in $\boldsymbol{K}$-space

$$\bar{R}_U = -\mathbb{1}$$

  - (ii) reflection on plane in $\boldsymbol{K}$-space

$$\bar{R}_U = R_U^{\mathrm{T}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_U\,, \qquad \text{with} \quad R_U \in SO(3)$$

Criteria for CP invariance of the potential

Consider CP symmetries of potential

$$V = \tilde{\xi}^{\mathrm{T}} \tilde{\boldsymbol{K}} + \tilde{\boldsymbol{K}}^{\mathrm{T}} \tilde{E} \tilde{\boldsymbol{K}}, \qquad \text{notation:} \quad \tilde{\xi} = \begin{pmatrix} \xi_0 \\ \boldsymbol{\xi} \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} \eta_{00} & \boldsymbol{\eta}^{\mathrm{T}} \\ \boldsymbol{\eta} & E \end{pmatrix}$$

and vacuum solution $\tilde{\boldsymbol{K}} = (K_0, \boldsymbol{K})^{\mathrm{T}}$.

Theorem (Necessary and sufficient CP criteria)

	potential symmetric	*vacuum symmetric*
CP type (i)	$\boldsymbol{\eta} = \boldsymbol{\xi} = 0$	*always broken*
CP type (ii)	$(\boldsymbol{\xi} \times \boldsymbol{\eta})^{\mathrm{T}} E \boldsymbol{\xi} = 0,$ $(\boldsymbol{\xi} \times \boldsymbol{\eta})^{\mathrm{T}} E \boldsymbol{\eta} = 0,$ $(\boldsymbol{\xi} \times (E\boldsymbol{\xi}))^{\mathrm{T}} E^2 \boldsymbol{\xi} = 0,$ $(\boldsymbol{\eta} \times (E\boldsymbol{\eta}))^{\mathrm{T}} E^2 \boldsymbol{\eta} = 0.$	$(\boldsymbol{\xi} \times \boldsymbol{\eta}))^{\mathrm{T}} \boldsymbol{K} = 0,$ $(\boldsymbol{\xi} \times (E\boldsymbol{\xi}))^{\mathrm{T}} \boldsymbol{K} = 0,$ $(\boldsymbol{\eta} \times (E\boldsymbol{\eta}))^{\mathrm{T}} \boldsymbol{K} = 0.$

note: basis independence, constructive proof
(type (ii) criteria equiv. to: Gunion, Haber; Nishi)

CP type (i) symmetric model

- general potential in basis where $E = \text{diag}(\mu_1, \mu_2, \mu_3)$ (μ_a real param.):

$$V = \xi_0 K_0 + \eta_{00} K_0^2 + \mu_1 K_1^2 + \mu_2 K_2^2 + \mu_3 K_3^2$$

- stability, required EWSB, no massless charged Higgses $\Leftrightarrow$

$$\eta_{00} > 0\,, \qquad \xi_0 < 0\,, \qquad \mu_3 < 0\,, \qquad \mu_a + \eta_{00} > 0 \quad \text{for } a = 1, 2, 3$$

- charged Higgs mass

$$m_{H^\pm}^2 = -2\mu_3 v_0^2 \qquad \text{with} \quad v_0^2 = \frac{-\xi_0}{\eta_{00} + \mu_3}$$

- mass matrix for neutral Higgses (ρ', h', h'')

$$\mathcal{M}_{\text{neutral}}^2 = 2 \begin{pmatrix} -\xi_0 & 0 & 0 \\ 0 & v_0^2(\mu_1 - \mu_3) & 0 \\ 0 & 0 & v_0^2(\mu_2 - \mu_3) \end{pmatrix}$$

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- CP type (i) symmetric Higgs-fermion coupling requires at least two fermion generations
- extending full symmetry of Higgs potential to Lagrangian: further restrictions

Layout

1. General two-Higgs-doublet model
   - THDM potential via gauge invariant functions
   - Minima
   - CP symmetries

2. NMSSM
   - NMSSM Potential
   - Gröbner bases
   - Minima

NMSSM Higgs potential

- motivation:

  MSSM: μ-problem $\quad W^H_{\text{MSSM}} = \mu\, \hat{H}_u \hat{H}_d \quad$ but: $\mu = \mathcal{O}(m_{\text{SUSY}})$

  NMSSM: scale via singlet VEV $\quad W^H_{\text{NMSSM}} = \lambda \hat{S}\, \hat{H}_u \hat{H}_d + \frac{1}{3}\kappa\, \hat{S}^3$

  (Fayet; Drees; Ellis, Gunion, Haber, Roszkowski, Zwirner)

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- scalar Higgs fields $\qquad H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}, \quad H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}, \quad S$
- Higgs potential

$$V = V_{\text{F}} + V_{\text{D}} + V_{\text{soft}}$$

  with

$$\begin{aligned} V_{\text{F}} =& |\lambda S|^2 (|H_u|^2 + |H_d|^2) + |\lambda H_u H_d + \kappa S^2|^2, \\ V_{\text{D}} =& \frac{1}{8}\bar{g}^2 (|H_d|^2 - |H_u|^2)^2 + \frac{1}{2} g^2 |H_u^\dagger H_d|^2, \\ V_{\text{soft}} =& m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 + m_S^2 |S|^2 + [\lambda A_\lambda S H_u H_d + \frac{1}{3}\kappa A_\kappa S^3 + \text{h.c.}]. \end{aligned}$$

- features:
  - physical Higgses: 5 neutral (3 “scalar” + 2 “pseudo-scalar”), 1 charged pair
  - explicite / spontaneous CP violation
  - bounded from below ($\lambda, \kappa \neq 0$) ✓
- *global minimum ?* here: algebraic solution

Towards the stationary points

- translate to same-hypercharge convention

$$\varphi_1^\alpha = -\epsilon_{\alpha\beta}(H_d^\beta)^*,$$
$$\varphi_2^\alpha = H_u^\alpha,$$

- use gauge invariant functions for doublets, $S = S_{re} + iS_{im}$ for singlet

$$\begin{aligned}
V_\mathrm{F} &= \frac{1}{4}|\lambda|^2 \left(K_1^2 + K_2^2 + 4K_0(S_{re}^2 + S_{im}^2)\right) + |\kappa|^2(S_{re}^2 + S_{im}^2)^2 \\
&\quad - \mathrm{Re}(\lambda\kappa^*) \left(K_1(S_{re}^2 - S_{im}^2) + 2K_2 S_{re} S_{im}\right) \\
&\quad + \mathrm{Im}(\lambda\kappa^*) \left(K_2(S_{re}^2 - S_{im}^2) - 2K_1 S_{re} S_{im}\right), \\
V_\mathrm{D} &= \frac{1}{8}\bar{g}^2 K_3^2 + \frac{1}{8} g^2 \left(K_0^2 - K_1^2 - K_2^2 - K_3^2\right), \\
V_\mathrm{soft} &= \frac{1}{2} m_{H_u}^2 (K_0 - K_3) + \frac{1}{2} m_{H_d}^2 (K_0 + K_3) + m_S^2 (S_{re}^2 + S_{im}^2) \\
&\quad - \mathrm{Re}(\lambda A_\lambda)(K_1 S_{re} - K_2 S_{im}) + \frac{2}{3}\mathrm{Re}(\kappa A_\kappa)\left(S_{re}^3 - 3S_{re}S_{im}^2\right) \\
&\quad + \mathrm{Im}(\lambda A_\lambda)(K_2 S_{re} + K_1 S_{im}) + \frac{2}{3}\mathrm{Im}(\kappa A_\kappa)\left(S_{im}^3 - 3S_{re}^2 S_{im}\right).
\end{aligned}$$

NMSSM stationarity conditions

- three classes of stationary points occur:

non-breaking : $K_0 = 0, \;\; \boldsymbol{K} = 0,$
$$\nabla V(S_{re}, S_{im}) = 0$$

full breaking : $K_0 > 0, \;\; K_0^2 - \boldsymbol{K}^2 > 0,$
$$\nabla V(K_0, K_1, K_2, K_3, S_{re}, S_{im}) = 0$$

partial breaking : $K_0 > 0, \;\; K_0^2 - \boldsymbol{K}^2 = 0,$
$$\nabla \left[V(K_0, K_1, K_2, K_3, S_{re}, S_{im}) - u(K_0^2 - \boldsymbol{K}^2) \right] = 0$$

NMSSM stationarity conditions

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$$\text{non-breaking}: \quad K_0 = 0, \quad \boldsymbol{K} = 0, \quad \nabla V(S_{re}, S_{im}) = 0$$

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$$\text{partial breaking}: \quad K_0 > 0, \quad K_0^2 - \boldsymbol{K}^2 = 0, \quad \nabla \left[V(K_0, K_1, K_2, K_3, S_{re}, S_{im}) - u(K_0^2 - \boldsymbol{K}^2) \right] = 0$$

- example for stat. equations (full breaking case):

$$27S_i^3 + 27S_iS_r^2 - 18K_2S_r + 18000S_iS_r + 24K_0S_i + 18K_1S_i + 5680K_2 - 2857800S_i = 0$$
$$27S_r^3 + 27S_i^2S_r - 18K_2S_i + 9000S_i^2 - 9000S_r^2 + 24K_0S_r - 18K_1S_r - 2857800S_r - 5680K_1 = 0$$
$$32526S_i^2 - 32526S_r^2 - 6291K_1 - 20527520S_r = 0$$
$$-65052S_iS_r - 6291K_2 + 20527520S_i = 0$$
$$29K_3 + 95977884 = 0$$
$$14456S_i^2 + 14456S_r^2 + 9325K_0 + 6716393125 = 0$$

- *How to solve this polynomial system of equations ?*
  $\Rightarrow$ Gröbner bases (“simpler” standard form: separation of variables)

Solving a linear system of equations

- System to solve

$$7x + y = 17 \tag{1}$$
$$x - y = -1 \tag{2}$$

(1) Ordering of variables (x, y)

$$\begin{pmatrix} 7 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 17 \\ -1 \end{pmatrix} = 0 \tag{3}$$

(2) Reduction
$row2 \rightarrow 7 * row2 - row1$:

$$\begin{pmatrix} 7 & 1 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 17 \\ -24 \end{pmatrix} = 0 \tag{4}$$

triangular form ("almost decoupled")

(3) Iterative solving

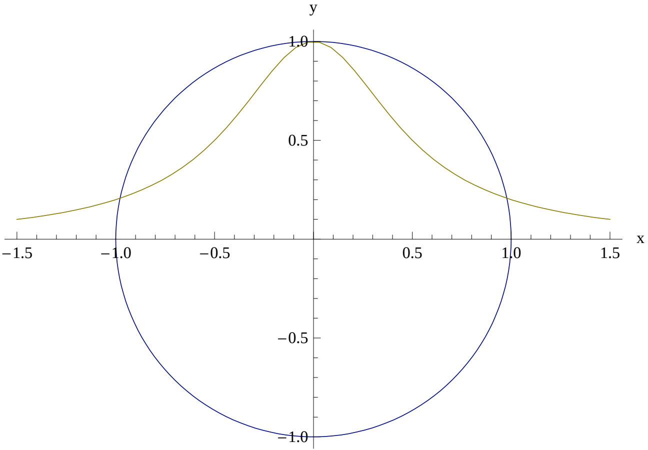
$$y = \frac{1}{8}(24) = 3 \tag{5}$$
$$x = \frac{1}{7}(17 - y) = \frac{1}{7}(14) = 2 \tag{6}$$

Nonlinear Multivariate Example
The task

Determine intersection of:

$$x^2 + y^2 = 1\,,$$

$$y = \frac{1}{1+4x^2}$$



Nonlinear Multivariate Example

First try: as for linear equations

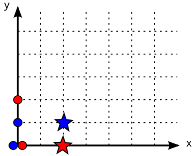
- System to solve:

$$x^2 + y^2 = 1$$
$$y = 1/(1 + 4x^2)$$

(1) Ordering: lexicographic (Lex) for $\mathbb{R}[x, y]$

$x^m \succ_{\text{lex}} y^n \ \forall m > 0$, monomials:
sort by powers of x, then by powers of y

$$x^2 + y^2 \quad - 1 = 0 \qquad (7)$$
$$4x^2 y \quad + y - 1 = 0 \qquad (8)$$



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$$x^2 + y^2 - 1 = 0 \qquad (7)$$
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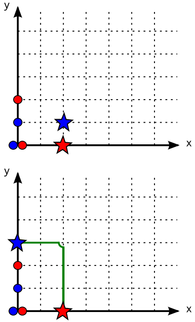
(2) Reduction: (8) $\rightarrow$ (8) $-$ $4y \cdot$ (7):

$$x^2 + y^2 - 1 = 0 \qquad (9)$$
$$-4y^3 + 5y - 1 = 0 \qquad (10)$$

equivalent to orig., but *"triangular form"*



Nonlinear Multivariate Example

First try: as for linear equations

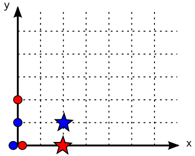
- System to solve: $$x^2 + y^2 = 1$$ $$y = 1/(1 + 4x^2)$$

(1) **Ordering**: lexicographic (Lex) for $\mathbb{R}[x, y]$

$x^m \succ_{\text{lex}} y^n \;\; \forall m > 0$, monomials:
sort by powers of x, then by powers of y

$$x^2 + y^2 \qquad - 1 = 0 \qquad (7)$$

$$4x^2y \qquad + y - 1 = 0 \qquad (8)$$



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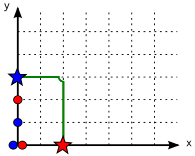


(3) **Iterative solving**

solve (10) for y:

$\Rightarrow \;\; y \in \{1, \; (-1 - \sqrt{2})/2, \; (-1 + \sqrt{2})/2\}$

plug solutions into (9) and solve for x:

$\Rightarrow \;\; (x, y) \in \{(-\sqrt{1 + 2\sqrt{2}}/2, (-1+\sqrt{2})/2), \; (0, 1), \; (\sqrt{1 + 2\sqrt{2}}/2, (-1+\sqrt{2})/2)\}$

(+ 3 complex non-real solutions)

Basic Definitions

Definition (Polynomial Ring)

A Polynomial Ring $K[x_1, \ldots, x_n] \equiv K[\mathbf{x}]$ is the set of all n-variate polynomials with variables $x_1, \ldots, x_n$ and coefficients in the field K.

Definition (Generated Ideal)

Let $F = \{f_1, \ldots, f_n\} \subset K[\mathbf{x}]$ be finite. then F generates an ideal defined by

$$I(F) \equiv \left\{ \sum_{f_i \in F} r_i \cdot f_i \,\middle|\, r_i \in K[\mathbf{x}],\ f_i \in F,\ i = 1, \ldots, n \right\}.$$

Notation: $\langle f_1, \ldots, f_n \rangle \equiv I(F)$

Definition (Reduction)

Let $f, p \in K[\mathbf{x}]$. We call f reducible modulo p, if for a term t of f there exists a term u with $\mathrm{LT}(p) \cdot u = t$, where LT denotes the leading term. Then we say, f reduces to h modulo p, where $h = f - \frac{\mathrm{Coefficient}(f,t)}{\mathrm{LC}(p)} \cdot u \cdot p$.

Nonlinear Multivariate Example

Second thought: first try as general recipe ?

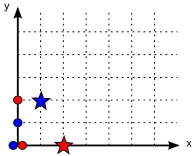
- Swap variables in system to solve:

$$x^2 + y^2 = 1$$
$$x = \frac{1}{1 + 4y^2}$$

(1) Ordering: lexicographic

generators of ideal $I = \langle f_1, f_2 \rangle$:

$$f_1 = x^2 \qquad + y^2 - 1 \qquad (= 0)$$
$$f_2 = \quad 4xy^2 + y \qquad - 1 \qquad (= 0)$$



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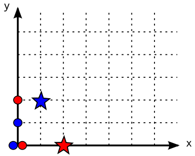

(2) Reduction: *not possible !*

Way out:

idea : provide missing leading terms for reductions

recipe : adjoin "S-polynomials" = differences of polys with LTs chopped off (Buchberger, 1965)

note : leads to *Gröbner bases*, "works always"

Nonlinear Multivariate Example

"Unlimited recipe"

- $I = \langle f_1, f_2 \rangle$, consider

$$\begin{aligned} spol(f_1, f_2) &= 4y^2 \cdot f_1 - x \cdot f_2 \\ &= 4y^2(x^2 + y^2 - 1) \\ &\quad - x(4xy^2 + x - 1) \\ &= -x^2 + x + 4y^4 - 4y^2 =: f_3 \end{aligned}$$

Nonlinear Multivariate Example

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- adjoin f_3 to generators $I = \langle f_1, f_2, f_3 \rangle$

$$\begin{aligned} f_1 &= x^2 + y^2 - 1 \\ f_2 &= 4xy^2 + x - 1 \\ f_3 &= -x^2 + x + 4y^4 - 4y^2 \end{aligned}$$

and interreduce

- f_3 reducible modulo f_1: $f_3' := f_3 + f_1 = x + 4y^4 - 3y^2 - 1$
- f_2 reducible modulo f_3': $f_2' := f_2 - f_3' = 4xy^2 - 4y^4 + 3y^2$
- f_2' reducible modulo f_3': $f_2'' := f_2' - 4y^2 f_3' = -16y^6 + 8y^4 + 7y^2$
- f_1 reduces in several steps to 0 modulo $\{f_2'', f_3'\}$

to $I = \langle f_3', f_2'' \rangle$

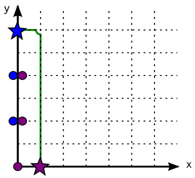
Nonlinear Multivariate Example

"Unlimited recipe" (cont.)

- we arrive at $I = \langle f_3', f_2'' \rangle$

  with

$$f_3' = x + 4y^4 - 3y^2 - 1$$
$$f_2'' = -16y^6 + 8y^4 + 7y^2$$



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Nonlinear Multivariate Example
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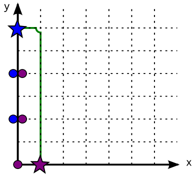


- $spol(f_3', f_2'')$ reduces to 0 modulo $\{f_3', f_2''\}$
- we found the reduced Gröbner basis $G = \{f_3', f_2''\}$

Gröbner Basis Definition

Definition (S-polynomial)

For $g_1, g_2 \in K[\mathbf{x}]$ the S-polynomial of g_1 and g_2 is defined as

$$\text{spol}(g_1, g_2) \equiv \text{LC}(g_2)\frac{\text{lcm}\left(\text{LT}(g_1), \text{LT}(g_2)\right)}{\text{LT}(g_1)}g_1 - \text{LC}(g_1)\frac{\text{lcm}\left(\text{LT}(g_1), \text{LT}(g_2)\right)}{\text{LT}(g_2)}g_2,$$

where lcm denotes the least common multiple.

Definition (Gröbner basis)

$G \subset K[\mathbf{x}]$ is called Gröbner Basis, if for all $f_1, f_2 \in G$

$$\text{spol}(f_1, f_2) \text{ reduces to } 0 \mod G$$

Theorem (LT of Gröbner basis)

$G = \{g_1, \ldots, g_n\} \subset K[\mathbf{x}]$ is a Gröbner Basis of the ideal $I \subset K[\mathbf{x}]$, iff

$$\langle \text{LT}(g_1), \ldots, \text{LT}(g_n)\rangle = \langle \text{LT}(I)\rangle$$

Algorithm (Buchberger)

For a given finite set $F \subset K[\mathbf{x}]$ *determine the Gröbner basis* $G \subset K[\mathbf{x}]$ *with* $I(F) = I(G)$. *Here,* $\mathrm{normf}(h, G)$ *denotes a fully reduced form of h w.r.t. to all members of G.*

$G := F$

$B := \{\{g_1, g_2\} | g_1, g_2 \in G \text{ with } g_1 \neq g_2\}$

while $B \neq \emptyset$ **do**

  choose $\{g_1, g_2\}$ *from* B

  $B := B \backslash \{\{g_1, g_2\}\}$

  $h := \mathrm{spol}(g_1, g_2)$

  $h' := \mathrm{normf}(h, G)$

  if $h' \neq 0$ **then**

    $B := B \cup \{\{g, h'\} | g \in G\}$

    $G := G \cup \{h'\}$

return G

Gröbner Basis Features

General highlights:

- general algorithmical method
- reduced GB unique
- full reduction modulo GB: independent of choice of reduction steps
- member of ideal $\Leftrightarrow$ reducable to 0

System solving:

- reduced Lex-GB has step-triangular form for 0-dim ideals
- system not solvable $\Leftrightarrow$ “1 is in GB” (reduces to 0)
- much more at purely algebraic level:
  dimension of solution space, number of solutions for dim=0, . . .
  ($\Rightarrow$ guarantee to find all minima)

Radical Ideals and the Shape Lemma

Definition (Radical)

Given an ideal $I \subset K[\mathbf{x}]$ its radical $\sqrt{I} \subset K[\mathbf{x}]$ is defined by

$$\sqrt{I} \equiv \{ f \in K[\mathbf{x}] \quad | \quad f^i \in I \text{ for some } i \geq 0 \}$$

Theorem (Shape Lemma)

Let K be a perfect field, $I \subseteq K[\mathbf{x}]$ a zero-dim. radical ideal in normal x_n-position, $g_n \in K[x_n]$ the monic generator of the elimination ideal $I \cap K[x_n]$ and $d = \deg(g_n)$.

- *The reduced Gröbner basis of I w.r.t. Lex-ordering is of the form*

$$\begin{array}{llllll} \{ & x_1 & & & & -g_1(x_n), \\ & & x_2 & & & -g_2(x_n), \\ & & & \ldots & & \\ & & & & x_{n-1} & -g_{n-1}(x_n), \\ & & & & & g_n(x_n) \quad \} \end{array}$$

  where $g_1, \ldots, g_{n-1} \in K[x_n]$
- *The polynomial g_n has d distinct zeros $a_1, \ldots, a_d \in \bar{K}$, and the set of zeros of I is*

$$\mathcal{Z}(I) = \{ \, (g_1(a_i), \ldots, g_{n-1}(a_i), a_i) \quad | \quad i = 1, \ldots, d \, \}$$

Real life issues

Run-time problems with presented recipe:

- original Buchberger algorithm: $\exp(n^2)$ in worst case
- computation of $\sqrt{I}$ expensive
- direct computation of Lex-GB expensive

(Partial) solutions to above problems used here:

- improved GB-algorithm: variant of Faugère's F4
- decompose GB in triangular systems (branches) a la Moeller, Hillebrand
- compute first Deg-GB, convert to Lex-GB with lin. algebra FGLM
- state-of-the-art implementations available in open source `Singular` CAS

NMSSM choice of parameters

- potential $V = V_F + V_D + V_{soft}$ with

$$V_F = |\lambda S|^2(|H_u|^2 + |H_d|^2) + |\lambda H_u H_d + \kappa S^2|^2,$$

$$V_D = \frac{1}{8}\bar{g}^2(|H_d|^2 - |H_u|^2)^2 + \frac{1}{2}g^2|H_u^\dagger H_d|^2,$$

$$V_{soft} = m_{H_u}^2|H_u|^2 + m_{H_d}^2|H_d|^2 + m_S^2|S|^2 + [\lambda A_\lambda S H_u H_d + \frac{1}{3}\kappa A_\kappa S^3 + \text{h.c.}].$$

- denote VEVs by

$$\left\langle H_d^0 \right\rangle = \frac{v_d}{\sqrt{2}}, \qquad \left\langle H_u^0 \right\rangle = e^{i\varphi_u}\,\frac{v_u}{\sqrt{2}}, \qquad \langle S \rangle = e^{i\varphi_S}\, v_s$$

- reparametrise potential using stat. cond. wrt fields
- parameters: gauge couplings, electroweak scale $v \equiv \sqrt{v_u^2 + v_d^2}$ and

$$\lambda,\ \kappa,\ |A_\kappa|\,,\ \tan\beta,\ v_S,\ m_{H^\pm},\ \text{sign}(R_\kappa),\ \delta_{EDM},\ \delta'_\kappa$$

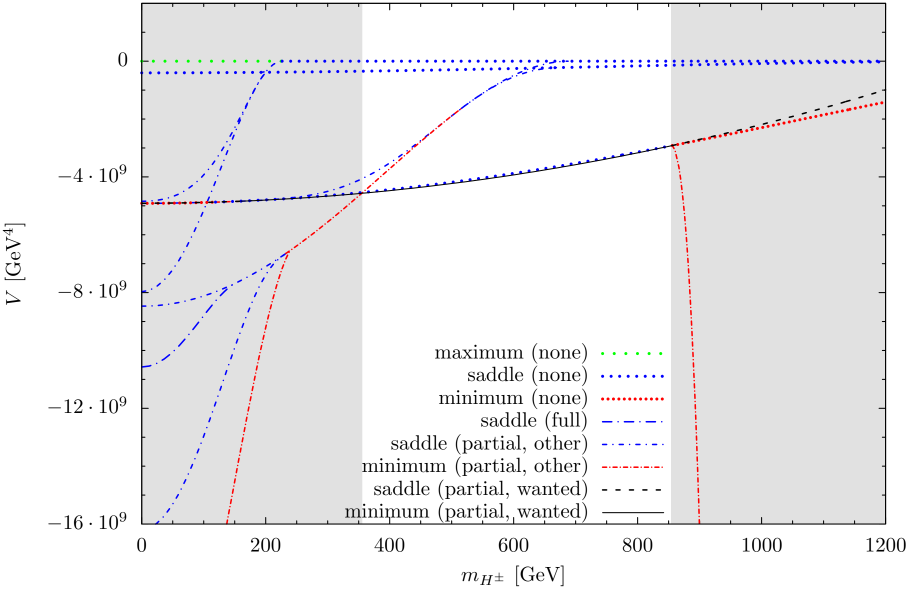
  where

$$\tan\beta \equiv \frac{v_u}{v_d}, \quad R_\kappa \equiv \text{Re}(\kappa A_\kappa e^{i3\varphi_S}), \quad \delta_{EDM} \equiv \delta_\lambda + \delta_u + \delta_S, \quad \delta'_\kappa \equiv \delta_\kappa + 3\delta_S$$

  (compatible to: Ellwanger; Funakubo, Tao)

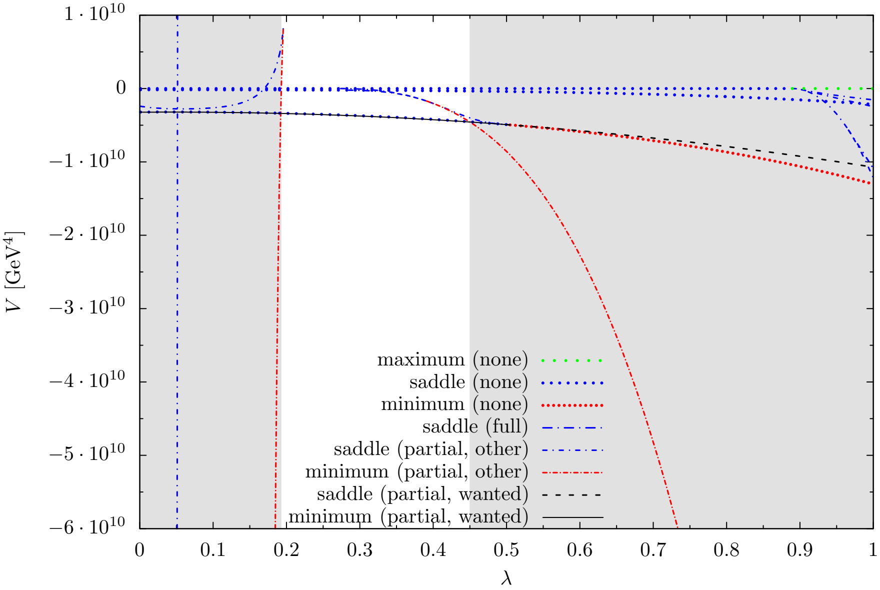
- “scalar”-“pseudo-scalar”-mixings in mass matrices $\propto \sin(\delta_{EDM} - \delta'_\kappa)$

NMSSM stationary points part 1



($\lambda = 0.4,\ \kappa = 0.3,\ |A_\kappa| = 200$ GeV, $\tan\beta = 3,\ v_S = 3v,$ sign $R_\kappa = -,\ \delta_{\text{EDM}} = 0,\ \delta'_\kappa = 0$)

NMSSM stationary points part 2



($\kappa = 0.3,\ |A_\kappa| = 200$ GeV$,\ \tan\beta = 3,\ v_S = 3v,\ m_{H^\pm} = 2v,\ \text{sign}\,R_\kappa = -,\ \delta_{\text{EDM}} = 0,\ \delta'_\kappa = 0$)

Results

Presented method asserts at the algebraical level:

- all minima are found (incl. CP viol.)

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- all minima are found (incl. CP viol.)

We find non-fine-tuned cases where

- wanted vacuum with “decent” pos. mass^2 $\neq$ global minimum

Some of these cases need special attention, e.g.:

- unwanted global minimum no “usual suspect” (all/one doublet zero, fully breaking)
- unwanted global minimum for large fields
- narrow valleys in potential

Summary

General THDM:

- gauge invariant functions simplify access to structure:
  - reduction of powers, no gauge d.o.f.
- simple flavour basis indep. statements
  - criteria for and easy access to global minimum
  - criteria for CP violation
  - new type of CP symmetry requires more than one fermion generation

NMSSM:

- Gröbner bases powerful tool for multivariate polynomial problems
- mathematically solid determination of global minimum at tree-level
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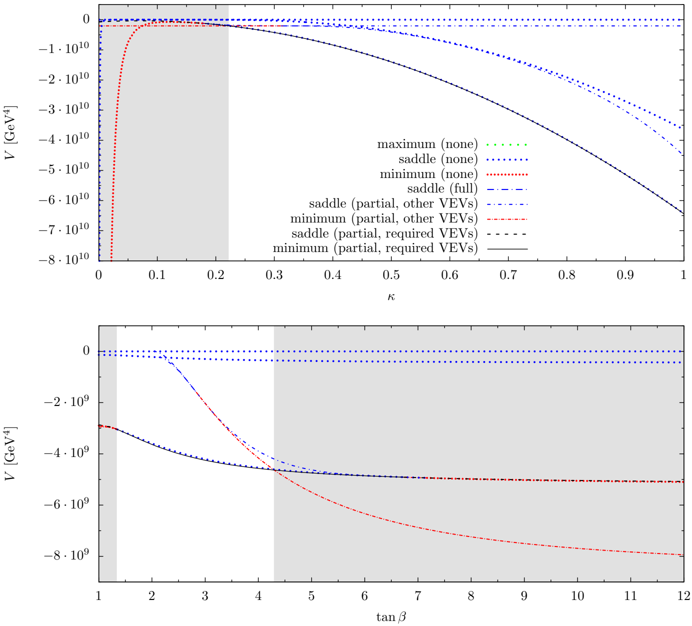
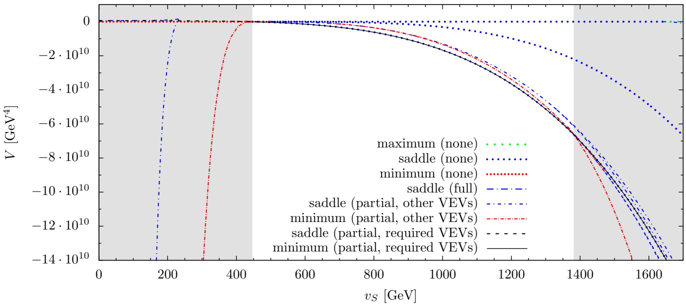
Uncovered + outlook:

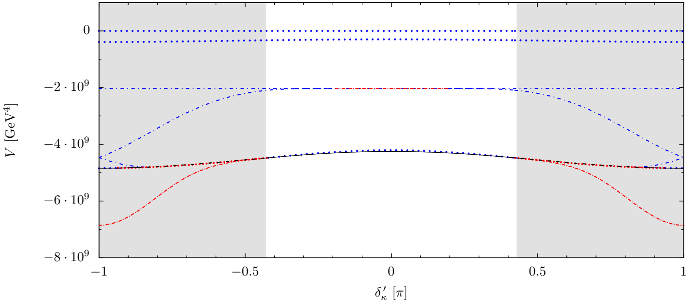
- criteria for positivity
- raditative corrections

NMSSM stationary points part 3



NMSSM stationary points part 4





"Almost general" THDM

- potential

$$V(\varphi_1,\varphi_2) = m_{11}^2 \varphi_1^\dagger\varphi_1 + m_{22}^2 \varphi_2^\dagger\varphi_2 - \left[m_{12}^2 \varphi_1^\dagger\varphi_2 + h.c. \right]$$
$$+\frac{1}{2}\lambda_1(\varphi_1^\dagger\varphi_1)^2+\frac{1}{2}\lambda_2(\varphi_2^\dagger\varphi_2)^2+\lambda_3(\varphi_1^\dagger\varphi_1)(\varphi_2^\dagger\varphi_2)+\lambda_4(\varphi_1^\dagger\varphi_2)(\varphi_2^\dagger\varphi_1)+\left[\frac{1}{2}\lambda_5(\varphi_1^\dagger\varphi_2)^2+h.c.\right], \tag{11}$$

  breaks $\varphi_1 \longrightarrow -\varphi_1$ only softly

- relation to our parameters:

$$\xi_0 = \frac{1}{2}(m_{11}^2 + m_{22}^2), \qquad \eta_{00} = \frac{1}{8}(\lambda_1+\lambda_2+2\lambda_3),$$

$$\boldsymbol{\xi} = \begin{pmatrix} -\operatorname{Re} m_{12}^2 \\ \operatorname{Im} m_{12}^2 \\ \frac{1}{2}\left(m_{11}^2 - m_{22}^2\right) \end{pmatrix}, \qquad \boldsymbol{\eta} = \frac{1}{8}\begin{pmatrix} 0 \\ 0 \\ \lambda_1-\lambda_2 \end{pmatrix},$$

$$E = \frac{1}{4}\begin{pmatrix} \lambda_4 + \operatorname{Re}\lambda_5 & -\operatorname{Im}\lambda_5 & 0 \\ -\operatorname{Im}\lambda_5 & \lambda_4 - \operatorname{Re}\lambda_5 & 0 \\ 0 & 0 & \frac{1}{2}(\lambda_1+\lambda_2-2\lambda_3) \end{pmatrix}$$

"Almost general" THDM cont.

- positivity by quartic terms:

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad \sqrt{\lambda_1 \lambda_2} + \lambda_3 > \max(0, |\lambda_5| - \lambda_4).$$

- $\mathrm{CP}_g^{(ii)}$ invariance of potential iff:

$$(\lambda_1 - \lambda_2)\ \mathrm{Im}\left((m_{12}^2)^2 \lambda_5^*\right) = 0\,,$$

$$\left[(\lambda_1 + \lambda_2 - 2(\lambda_3 + \lambda_4))^2 - 4\,|\lambda_5|^2\right](m_{11}^2 - m_{22}^2)\ \mathrm{Im}\left((m_{12}^2)^2 \lambda_5^*\right) = 0$$

- $\mathrm{CP}_g^{(ii)}$ invariance of potential and vacuum iff at least one of

$$v_1 v_2 \left[\cos(2\zeta)\,\mathrm{Im}\,\lambda_5 + \sin(2\zeta)\,\mathrm{Re}\,\lambda_5\right] = 0$$

  or

$$\lambda_1 = \lambda_2, \qquad (v_1^2 - v_2^2)\left[(\lambda_3 + \lambda_4 - \lambda_1)^2 - |\lambda_5|^2\right] = 0$$

  fulfilled (ζ complex phase of $\langle \varphi_2^0 \rangle$)