Worldlines and Bound States:
A New Approach to the Relativistic Binding Problem

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PSI, October 14, 2005

with

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Outline:

1. Introduction: a brief history of wordlines and a polaron primer
2. The relativistic polaron: WC model
3. The relativistic bound-state problem
4. Results and discussion
5. Beyond the quenched approximation
6. Summary and outlook

1. Introduction

Quantum Mechanics:

$$< x_b \,|\, \exp\left[-i(t_b - t_a)\widehat{H}/\hbar\right] \,|\, x_a > \quad = \quad \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t)\; e^{iS[x(t)]/\hbar}$$

Heisenberg, Schrödinger, Dirac (1925 - 1927) — Dirac (1933), Feynman (1942)

operators, wavefunctions $\longleftrightarrow$ path integrals, trajectories

“ WAVES ” $\longleftrightarrow$ “ PARTICLES ”

Field Theory:

field operators $\widehat{\phi}(x)$, states $\longleftrightarrow$ worldlines $x_\mu(t)$

Jordan, Heisenberg, Pauli ($\sim$ 1930) — Feynman ($\sim$ 1950)

“ FIELDS ” $\longleftrightarrow$ “ PARTICLES ”

" second quantization " $\longleftrightarrow$ " first quantization "
Dyson (1949)

$\Downarrow$

wins !
(see textbooks)

renaissance ... from string theory (!) Bern & Kosower (1991)

Strassler (1992) showed how to derive the Bern-Kosower rules from the particle (worldline) representation of Quantum Field Theory

Advantages:

a) efficient way to calculate diagrams with many legs

b) new approximation methods for large couplings (as in the polaron problem)

Polaron = electron slowly moving through polarizable crystal
(e.g. NaCl)

model Hamiltonian H. Fröhlich (1954)

$$\hat{H} \sim \frac{1}{2}\hat{\mathbf{p}}^2 + \sum_k \hat{a}_k^\dagger \hat{a}_k + \sqrt{\alpha} \sum_k \frac{1}{|\mathbf{k}|} \left[\hat{a}_k^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} + h.c. \right]$$

α: dimensionless electron-phonon coupling constant.

Feynman (1955): phonons can be integrated out exactly in path integral

$$\Rightarrow S_{\text{eff}} \sim \int_0^\beta dt\, \frac{1}{2}\dot{\mathbf{x}}^2 + \alpha \int_0^\beta dt\, dt'\, e^{-|t-t'|} \int d^3k\, \frac{1}{\mathbf{k}^2} \exp\left[i\mathbf{k} \cdot \left(\mathbf{x}(t) - \mathbf{x}(t') \right) \right]$$

↑
Coulomb propagator

one-particle problem now, but with two-time (retarded) action !

variational principle for

$$Z(\beta) \;=\; \oint \mathcal{D}x\, e^{-S_{\rm eff}} = \oint \mathcal{D}x\, e^{-S_t} \cdot \underbrace{\frac{\oint \mathcal{D}x\ \exp\left(-S_t - (S_{\rm eff} - S_t)\right)}{\oint \mathcal{D}x\ \exp(-S_t)}}_{=:\langle e^{-\Delta S}\rangle} \quad \xrightarrow{\beta\to\infty} \quad e^{-\beta E_0}$$

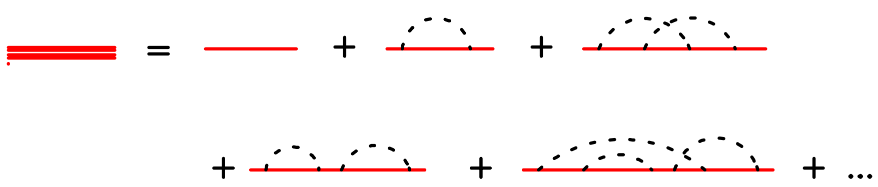
Jensen's inequality $\left\langle e^{-\Delta S}\right\rangle \ge e^{-\langle \Delta S\rangle} \implies$ ground state energy $E_0(\alpha)$ of polaron

trial action: $$S_t \;=\; \int_0^\beta dt\, \frac{\dot{x}^2}{2} + \int_0^\beta dt\, dt'\, f\left(|t-t'|\right)\,\left[\mathbf{x}(t) - \mathbf{x}(t')\right]^2$$

$\implies$ nonlinear variational eq. for retardation function $f(\sigma)$

best analytical method which works for **all** α !

sums (approximately) self-energy diagrams:



2. The relativistic polaron

Derivation of worldline description from ordinary QFT:

consider Wick-Cutkosky (WC) model for scalar "nucleon" Φ and "meson" χ
RR & A. Schreiber, Phys. Rev. **D53** (1996)

$$\mathcal{L} \;=\; \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}M_0^2\Phi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m^2\chi^2 - g\Phi^2\chi$$

generating functional for Green functions

$$Z[J,j] \;=\; \int \mathcal{D}\Phi\mathcal{D}\chi\, \exp\Big[\, i \int d^4x\, \left(\mathcal{L} + J\Phi + j\chi\right) \Big]$$

steps for calculation of 2-point function

$$G_2(x) \;:=\; \left\langle g.s. \left| \mathcal{T}\hat{\Phi}(x)\hat{\Phi}(0) \right| g.s. \right\rangle = \left. \frac{\delta^2 Z}{\delta J(x)\delta J(0)} \right|_{J=j=0}$$

1) integrate out "nucleon" (Gaussian integral !)

$$G_2(x) \;=\; \int \mathcal{D}\chi \, \det{}^{-1/2}\left(\mathcal{O}_\chi\right) \cdot \left\langle x | \mathcal{O}_\chi^{-1} | 0 \right\rangle \, \exp\left[i \int d^4x \, \mathcal{L}_0(\chi) \right]$$

$$\mathcal{O}_\chi \;=\; -\partial^2 - M_0^2 - 2g\chi$$

neglect det $\mathcal{O}_\chi$, i.e. pair production: **quenched approximation**

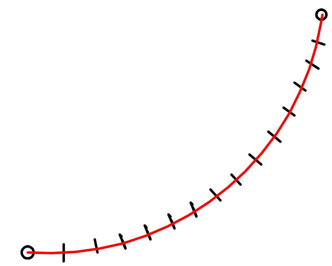
2) proper time (Schwinger) representation for particle propagator in the presence of meson field $\chi(x)$

$$\left\langle x | \mathcal{O}_\chi^{-1} | 0 \right\rangle \;=\; \frac{1}{2i} \int_0^\infty dT \, \left\langle x \left| \exp\left[\frac{i}{2} \left(-\partial^2 - M_0^2 - 2g\chi + i\epsilon \right) T \right] \right| 0 \right\rangle$$

3) write matrix element as (quantum mechanical) path integral

$$\left\langle x \left| \exp\left[\frac{i}{2} \left(-\partial^2 - 2g\chi \right) T \right] \right| 0 \right\rangle \;=\; \int_{x(0)=0}^{x(T)=x} \mathcal{D}x(t) \, \exp\left[i \int_0^T dt \, \left(-\frac{1}{2}\dot{x}^2 + g\chi(x) \right) \right]$$

over particle trajectory $x_\mu(t)$:



4) perform integration over meson field χ $\Rightarrow$ everything expressed in terms of particle coordinate !

$$G_2(x) = \int_0^\infty dT \, \exp\left[-\frac{iM_0^2}{2} T \right] \int_{x(0)=0}^{x(T)=x} \mathcal{D}x \, \exp\left\{ iS_{\text{eff}}[x] \right\}$$

with

$$S_{\text{eff}}[x] = \int_0^T dt \left(-\frac{1}{2}\dot{x}^2 \right) - \frac{g^2}{2} \int_0^T dt \, dt' \int \frac{d^4k}{(2\pi)^4} \frac{\exp\left\{ ik \cdot [x(t) - x(t')] \right\}}{k^2 - m^2 + i0}$$

↑
free meson propagator

effective, retarded action for the nucleon

Variational approximation for the Fourier transform $G_2(p)$:

apply Feynman-Jensen var. principle with **retarded**, quadratic trial action

$$S_t[x] \;=\; \int_0^T dt \left(-\frac{1}{2}\dot{x}^2\right) + \lambda\, p \cdot \int_0^T dt\, \dot{x} + \int_0^T dt\, dt'\, f(|t-t'|)\, \left[x(t) - x(t')\right]^2$$

propagator develops pole for $T \to \infty$:

$$G_2(p) \;\longrightarrow\; \frac{Z}{p^2 - M^2}$$

physical mass M determined by **Mano's eq.** Mano (1955)

$$\boxed{M_0^2 \;=\; M^2\left(2\lambda - \lambda^2\right) + 2\left\{\; \Omega[f] + V[f] \;\right\}}$$

↑ kinetic ↑ potential term

by construction the RHS is stationary w.r.t. variations of $f(\sigma)\,,\lambda$

$\Rightarrow$ nonlinear system of variational equations

Behaviour of retardation function $f(\sigma)$ for small σ :

$$f(\sigma) \longrightarrow \frac{\text{const.}}{\sigma^{d/2}}$$

$\mathbf{d=3}$ (polaron): no divergencies at all

$\mathbf{d=4}$ (WC model): no divergencies in variational eqs.,
only mass renormalization necessary:

$$M_1^2 \;=\; M_0^2 - \frac{g^2}{4\pi^2}\,\ln\left(\frac{\Lambda^2}{m^2}\right)$$

WC model is super-renormalizable !

renormalizable field theory (e.g. QED):

$$f(\sigma) \longrightarrow \frac{\text{const.}}{\sigma^{d/2+1}}$$

What do we learn from Mano's eq. ?

- dressing of "nucleon/electron" by cloud of "mesons/photons"
- Wick-Cutkosky model is un/meta-stable: interaction $g\Phi^2\chi \,\hat{=}\, \Phi^3$
  Baym (1960)

  not seen in perturbation theory but var. approach finds **critical coupling**

$$\alpha_{\text{crit}} = \frac{g^2_{\text{crit}}}{4\pi M^2}$$

  above which there are no real solutions of the var. eqs. anymore

  rough variational estimate for $m = 0$: take $f(\sigma) = 0 \implies$ var. eq. for λ can be solved

$$\lambda = \frac{1}{2}\left[1 + \sqrt{1 - \frac{4\alpha}{\pi}}\right] \Rightarrow \alpha_{\text{crit}} = \frac{\pi}{4} = 0.785 \text{ (numerically 0.641)}$$

  $\alpha > \alpha_{\text{crit}}$: can calculate **width/lifetime** of metastable "nucleon"

Extensions

- Processes with one external meson: non-perturbative calculation of form-factor
  Schreiber, RR & Alexandrou, Nucl. Phys. **A 601** (1996)

- Processes with two external mesons: meson-nucleon scattering
  Alexandrou, RR & Schreiber, Nucl. Phys. **A 628** (1998) ;
  deep-inelastic inclusive scattering
  Fettes & RR, Few-Body Syst. **24** (1998)

- Improved (anisotropic) trial actions
  Schreiber & RR, Eur. Phys. J. **C 25** (2002)

- quenched QED: non-perturbative expression for anomalous mass dimension of electron
  Alexandrou, RR & Schreiber, Phys. Rev. **D 62** (2000);
  Abraham-Lorentz-like equation for electron
  RR & Schreiber, Eur. Phys. J. **C 37** (2004)

3. Bound-state problem

"In the relativistic approach, bound states and resonances are identified by the occurrence of poles in Green functions. A simple extension of the Schrödinger equation is unfortunately not available, except in limiting cases ..."

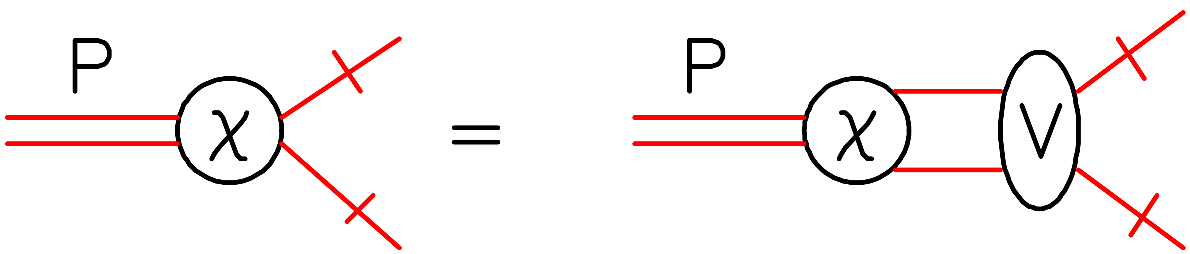
Itzykson & Zuber: Quantum Field Theory, p. 481

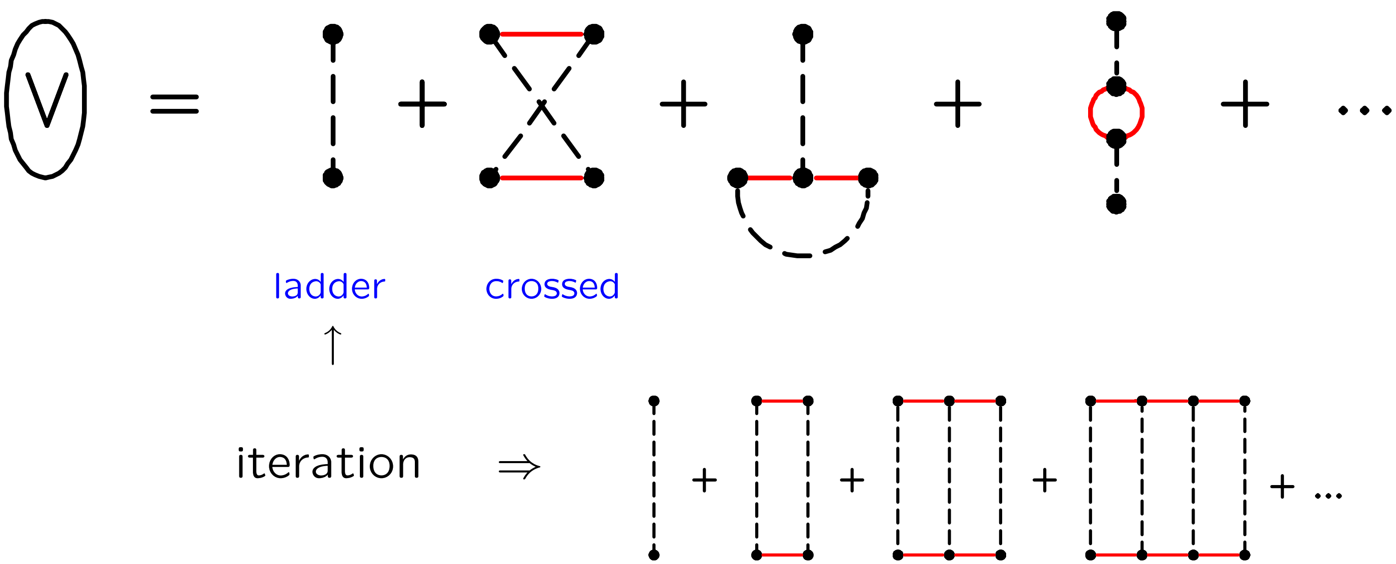
many approaches over the years:

- Bethe-Salpeter eq. (ladder approximation)
- 3-dimensional reductions (quasi-potential equations)
- perturbation theory (QED)
- effective field-theory methods (for weak coupling)
- light-cone methods
- lattice calculations (purely numerically)

⋮

Bethe-Salpeter equation:





New approach: worldline variational method

Barro-Bergflödt, RR & Stingl, Mod. Phys. Lett. **A 20** (2005)

applied to scalar WC-model for **two** particles of the same bare mass M_0

try to find an additional pole in the **correlator** or **polarization propagator** (special 4-point function)

$$\Pi(q) = \int d^4x \, e^{iq\cdot x} \left\langle \text{g.s.} \left| \mathcal{T} \left(\hat{\Phi}(x)\hat{\Phi}(x) \, \hat{\Phi}(0)\hat{\Phi}(0) \right) \right| \text{g.s.} \right\rangle \quad \text{for } q^2 < 4M^2$$

proceeding as before $\Rightarrow$ double worldline description
(in quenched approximation)

two proper times
↓

$$\Pi(q) = \int d^4x \, e^{iq\cdot x} \int_0^\infty dT_1 \, dT_2 \exp\left[-\frac{i}{2} M_0^2 \left(T_1 + T_2\right) \right] \cdot \int_{x_1(0)=0}^{x_1(T_1)=x} \mathcal{D}x_1 \int_{x_2(0)=0}^{x_2(T_2)=x} \mathcal{D}x_2 \, \exp\left\{ i \sum_{k=1}^{2} S_0[x_k] + i \, S_{\text{int}} \right\}$$

with

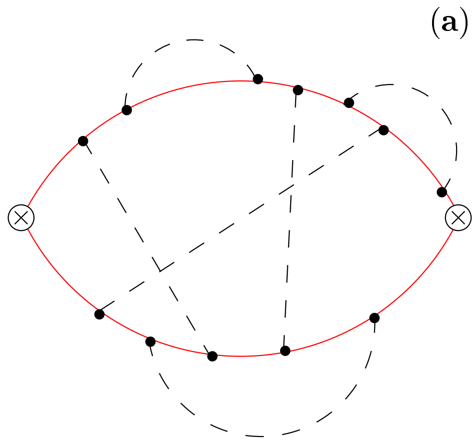
$$S_{\text{int}}[x_1, x_2] = -\frac{g^2}{2} \sum_{j,k=1}^{2} \int_0^{T_j} dt \int_0^{T_k} dt' \int \frac{d^4k}{(2\pi)^4} \frac{\exp\left[-ik \cdot (x_j(t) - x_k(t'))\right]}{k^2 - m^2 + i0}$$

↑
free meson propagator

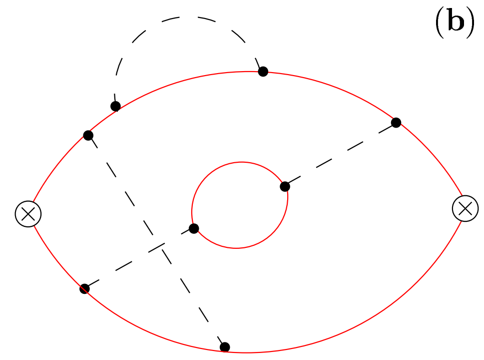
$j = k$: self-interactions
$j \neq k$: direct interaction between particle j and k

$\Big\}\Rightarrow$ vertex corrections

typical diagrams:



quenched



unquenched

Fourier expansion of paths

$$x_j(t) \sim x\frac{t}{T_j} + \sum_{k=1}^{\infty} a_k^{(j)} \sin\left(\frac{k\pi t}{T_j}\right)$$

$$\Longrightarrow S_0 \sim q\cdot x + \sum_{j=1}^{2}\left[\frac{x^2}{T_j} + \sum_{k=1}^{\infty} a_k^{(j)\,2}\right]$$

quadratic trial action:

$$S_t \sim \lambda q\cdot x + \sum_{j=1}^{2}\left[A_0\frac{x^2}{T_j} + \sum_{k=1}^{\infty} A_k\, a_k^{(j)\,2}\right] + \underbrace{\sum_{k=1}^{\infty} B_k\, a_k^{(1)}\cdot a_k^{(2)}}_{\text{coupling of particle 1 and 2}}$$

$\Pi(q)$ develops pole for $T = T_1 + T_2 \to \infty$ determined by

Mano's (bound-state) eq.

$$M_0^2 = \left(\frac{q}{2}\right)^2 (2\lambda - \lambda^2) - \Omega_{12} - 2\Big(V_{11}[A,B] + V_{12}[A,B]\Big)$$

Note:

a) V_{11} must be regularized ; divergent part can be absorbed in

$$M_1^2 = M_0^2 - \frac{g^2}{4\pi^2} \ln\left(\frac{\Lambda^2}{m^2}\right)$$

mass renormalization exactly as in one-body case !

b) V_{11}, V_{12} depend **both** on A, B: coupling of self-energy and direct interaction $\Rightarrow$ vertex corrections

c) all crossed diagrams are (approximately) taken into account: no ladder approximation, no spectator approximation, no 3-dimensional reduction

d) **disadvantage**: direct interaction is also approximated by (retarded) oscillator $\Longrightarrow$ not a very precise approximation , especially near threshold

4. Results and discussion

Weak coupling expansion for binding energy $(m = 0,\ V_{12} \to Z\, V_{12},\ \alpha \equiv \frac{g^2}{4\pi M^2})$

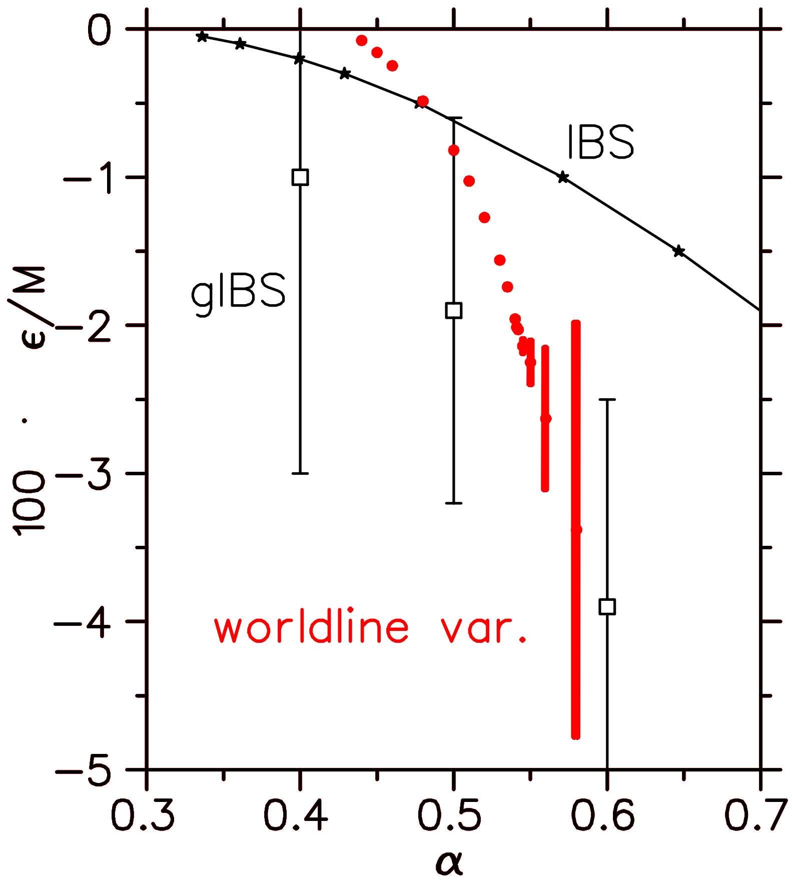
$$\frac{E_0}{M/2} = -b_2\,(Z\alpha)^2\left[1 + r_1\,\frac{\alpha}{\pi} + \ldots\right] - b_4\,(Z\alpha)^4\left[1 + \ldots\right] - \ldots$$

	var.	exact
b_2 :	$\frac{1}{\pi} = 0.318$	$\frac{1}{2}$ (Coulomb)
r_1 :	$\frac{7}{2} = 3.5$	4 (eff. field theory)
b_4 :	$\frac{1}{\pi^2} = 0.101$	$\frac{5}{32} = 0.156$ (Todorov's eq.)

smaller numerical coeffcients as expected from a variational calculation

Numerical solutions of var.eqs.: have found pole below $q^2 < 4M^2$

binding energy ϵ vs. coupling constant $\alpha = \frac{g^2}{4\pi M^2}$:



What about instability of WC model ?

shows up again, but at lower values of the coupling constant:
induced instability !

rough variational estimate for $m = 0$: take (harmonic-oscillator-like) ansatz

$$A_k = 1 + \frac{\omega^2}{2(k\pi/T)^2} \; , \quad B_k = \frac{\omega^2}{2(k\pi/T)^2}$$

$\Longrightarrow$ var. eqs. for λ and $\omega \ll M^2$ can be solved analytically:

$$\lambda^4 - \lambda^3 + \frac{\alpha}{\pi}\lambda^2 + \frac{1}{2\pi}(Z\alpha)^2 = 0$$

critical value of coupling constant now

$$\alpha_{\rm crit} \xrightarrow{Z \to 0} \frac{\pi}{4}\left[1 - \frac{\pi}{2}Z^2 + \ldots\right]$$

monotonically decreasing with Z

$$Z = 1 \quad \Longrightarrow \quad \alpha_{\rm crit} = 0.463 \qquad (0.785 \text{ for } Z = 0\,)$$

(0.54 from numerical solution of var. eqs.)

5. Beyond the quenched approximation

Up to now **vacuum polarization** (VP) terms had to be neglected because

1. Number of worldlines for heavy particles is conserved (physics)

2. Meson/photon field cannot be integrated out with VP effects included (mathematics)

example: scalar Wick-Cutkosky model with N nucleon species

functional integral over meson field χ contains determinant

$$D[\chi] \;=\; \left[\,\text{Det}\left(-\partial^2 - M_0^2 - 2g\chi\right)\,\right]^{-N/2}$$

How to include VP effects?

First possibility:

$$\text{Expand} \quad \frac{D[\chi]}{D[0]} \;=\; \exp\left\{-\frac{N}{2}\text{Tr}\ln\left[1 + \frac{2g}{\partial^2 + M_0^2}\chi\right]\right\}$$

to second order in the meson field ($M_0 \approx M$)

$$\frac{D[\chi]}{D[0]} \approx \exp\left\{ -\frac{N}{2}\mathrm{Tr}\left[2g \, \frac{1}{\partial^2 + M^2}\chi - \frac{(2g)^2}{2} \frac{1}{\partial^2 + M^2} \chi \frac{1}{\partial^2 + M^2} \chi + \ldots \right] \right\}$$

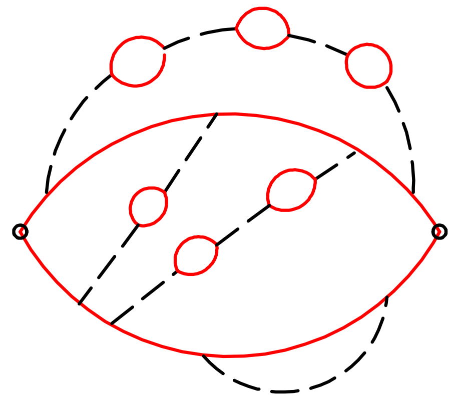
linear terms in χ: tadpoles (irrelevant)

quadratic terms in χ:

- meson mass renormalization
- modification of meson propagator by (renormalized) one-loop VP contribution
- no other change in interaction part of the n-particle worldline action

$$-\frac{g^2}{2} \sum_{i,j=1}^{n} \int_0^{T_i} dt \int_0^{T_j} dt' \int \frac{d^4k}{(2\pi)^4} \frac{\exp\left[-ik \cdot (x_i(t) - x_j(t')) \right]}{k^2 - m^2 - \pi_r(k^2)} \, .$$

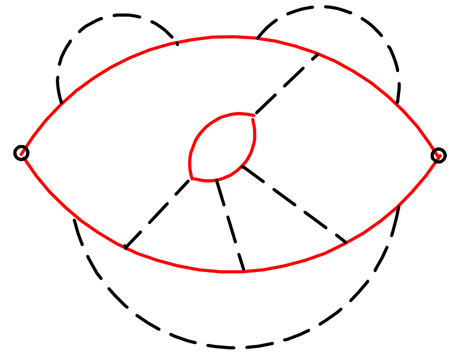
Contains VP insertions in all meson lines:



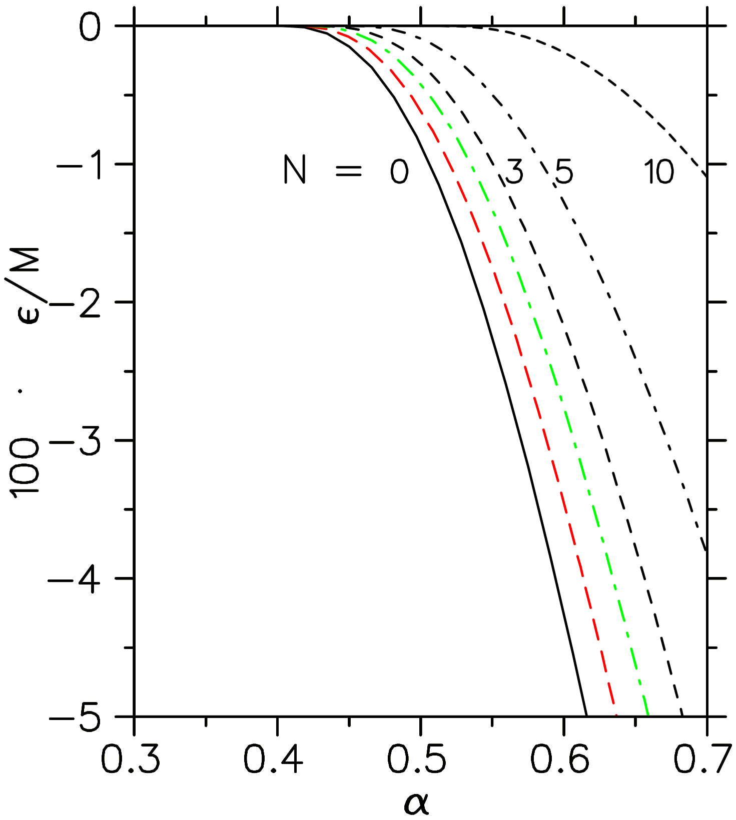
main effect : $\alpha \longrightarrow \alpha^\star = \dfrac{\alpha}{1 + \frac{\alpha}{6\pi} N}$

↑ number of flavors

but **no** interaction of pair-produced particles:



Binding energy vs. coupl. constant for different N :



Second possibility:

"hybrid" approach: apply variational principle for **both** meson field χ and worldline $x(t)$

as in the **linear polaron model** Bogoliubov (1978)

modified meson propagator
↓

$$S_t[x,\chi] \;=\; \text{quadratic in } x(t) \;+\; \text{quadratic in } \chi$$

$$+\int_0^T dt \int d^4k \;\ell(k^2)\, ik\cdot x(t)\,\chi(k)$$

↑
variational coupling function

solvable trial action, averages can be evaluated,

(numerical) results in progress

6. Summary and outlook

Worldline variational methods for field theories

- are successful due to reduction in number of variables:
  $\Phi(x), \chi(x) \quad \longrightarrow \quad x(t)$
- have been applied to scalar and fermionic theories, give gauge-invariant results
- offer a new approach to the relativistic bound-state problem which includes self-energy effects and vertex corrections
- can include vacuum-polarization effects consistently

Possible extensions:

- ◦ improved (anisotropic) trial action
- ◦ systematic second-order corrections to variational result
- ◦ inclusion of other internal degrees of freedom (color) in worldline formalism. QCD ?

  ⋮