Measurements, uncertainties and probabilistic inference/forecasting

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(A. Einstein)

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"Probability is good sense reduced to a calculus"
(S. Laplace)

What is measurement?

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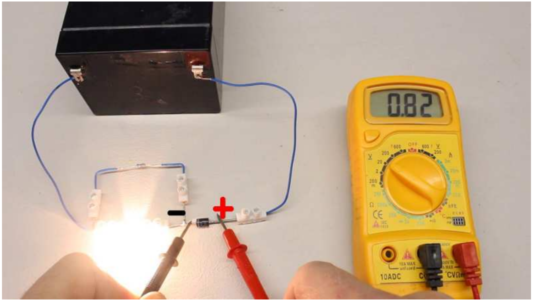

What is measurement?

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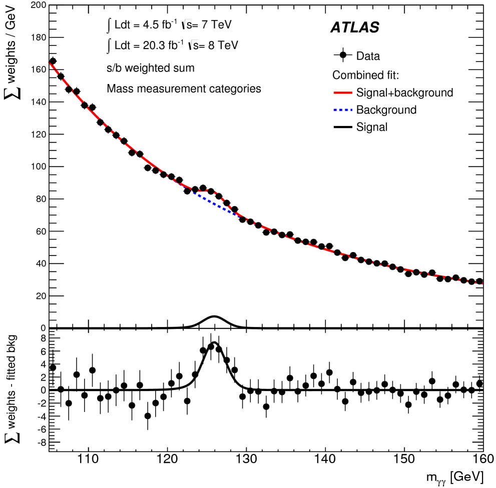
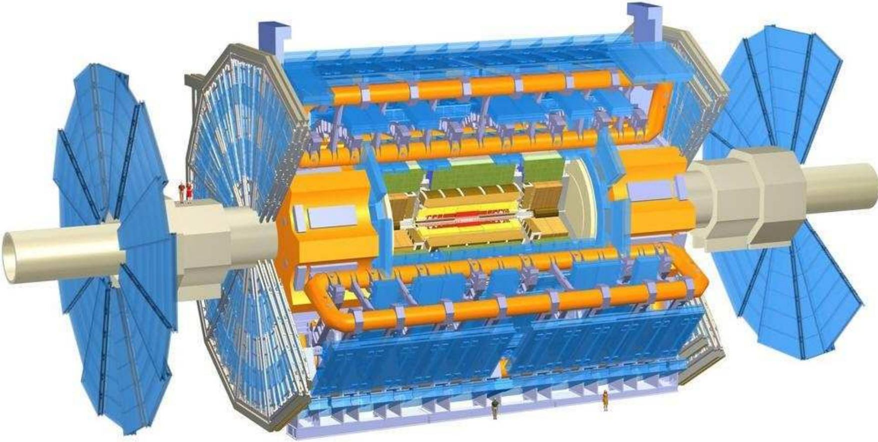
Higgs $\to \gamma\gamma$ (2012)



Two-photon *invariant mass*

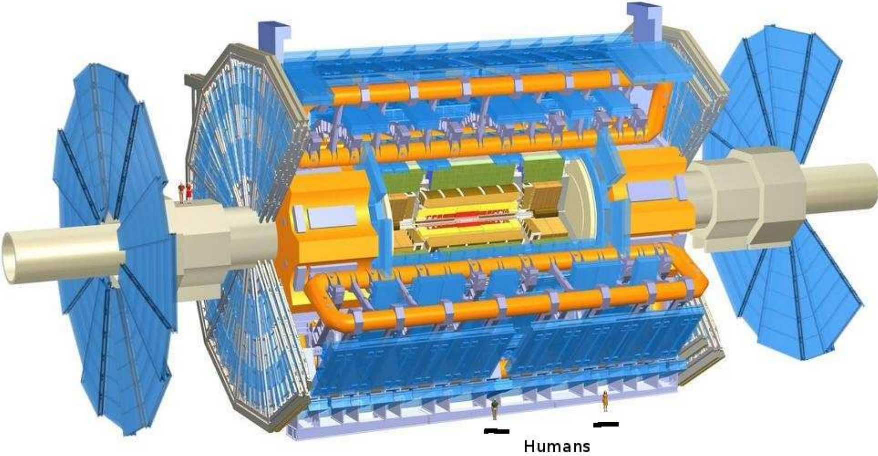
What is measurement?

ATLAS Experiment at LHC (CERN, Geneva)



What is measurement?

ATLAS Experiment at LHC [length: 46 m; ⌀ 25 m]



≈ 3000 km cables

≈ 7000 tonnes

≈ 100 millions electronic channels

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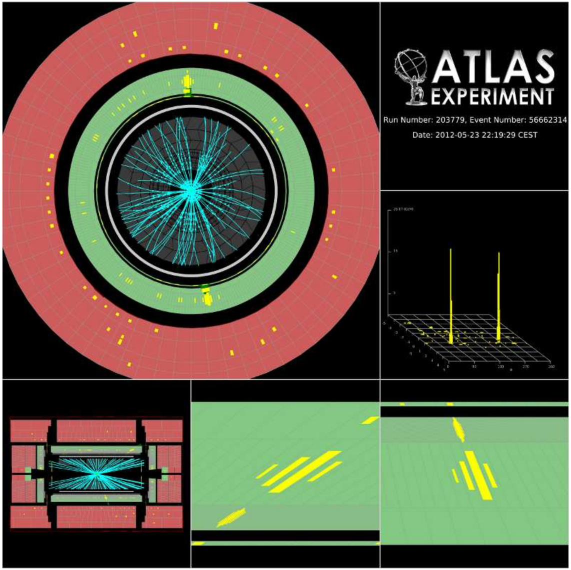

Two flashes of ‘light’ (2 γ’s) in a ‘noisy’ environment.

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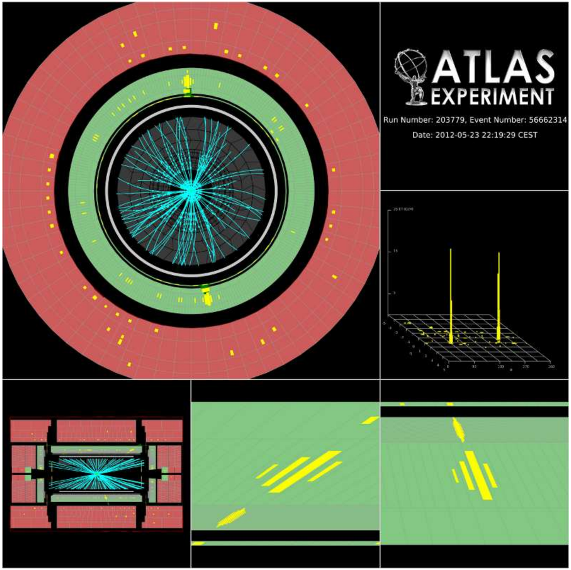

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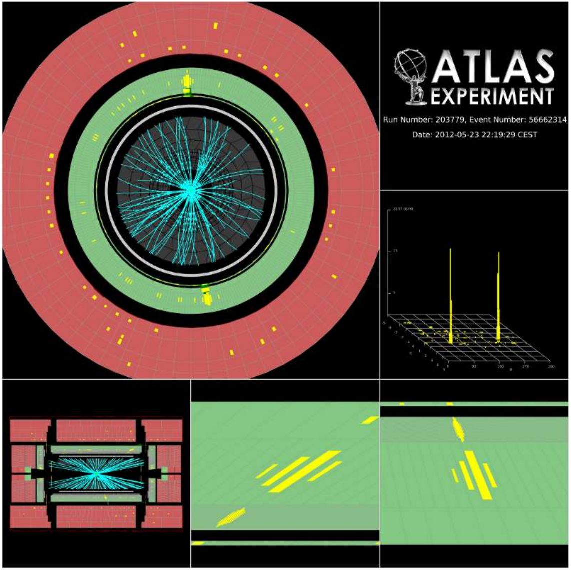

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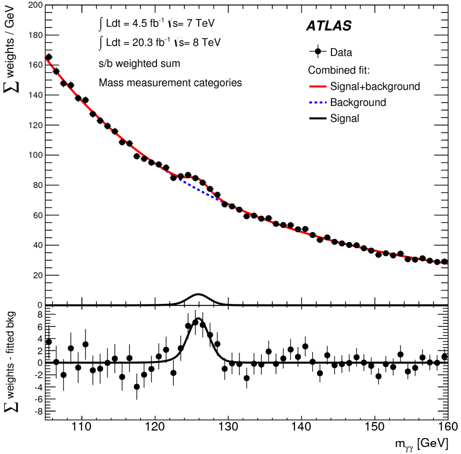
Higgs $\to \gamma\gamma$? Probably not. . .

What is measurement?

Higgs $\to \gamma\gamma$



$\Rightarrow \begin{cases} \text{Mass value} \\ \text{Production rate} \end{cases}$

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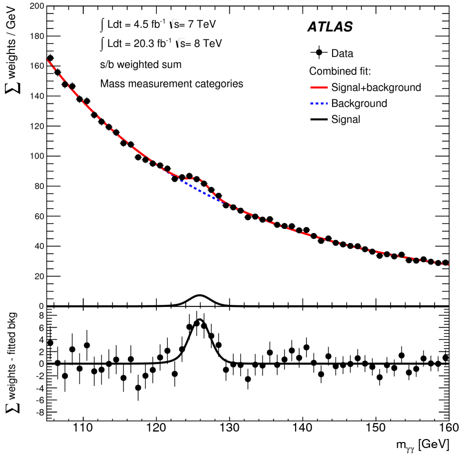

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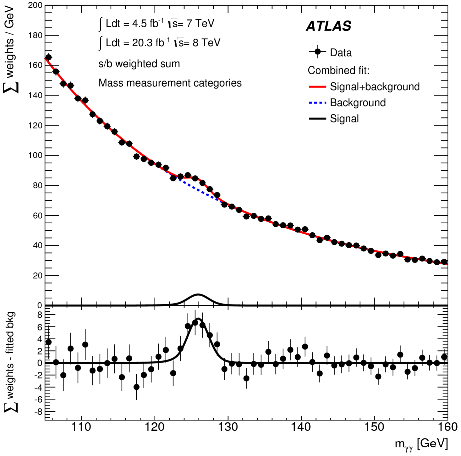

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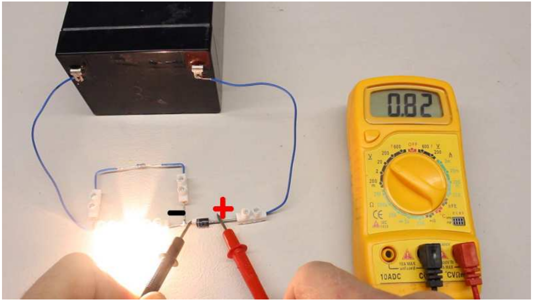
Quite indirect measurements of something we do not "see"!

Can we "see" physics quantities?

But, can we see our mass?

Can we “see” physics quantities?

. . . or a voltage?



Can we "see" physics quantities?

. . . or our blood pressure?

Can we “see” physics quantities?

Certainly not!

Can we “see” physics quantities?

Certainly not!

. . . although for some quantities we can have

a ‘vivid impression’ (in the David Hume’s sense)

Measuring a mass on a scale

Equilibrium:

$$\begin{aligned} {\color{red}m}g - k{\color{blue}\Delta x} &= 0 \\ {\color{blue}\Delta x} &\to {\color{blue}\theta} \to \text{scale reading} \end{aligned}$$

(with 'g' *gravitational acceleration;* 'k' spring constant.)

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From the reading to the value of the mass:

$$\text{scale reading} \xrightarrow[\text{given } g,\, k,\, \text{"etc."}\ldots]{} {\color{red}m}$$

Measuring a mass on a balance

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Certainly not to watch our weight

But think about it!

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$$\text{scale reading} \xrightarrow[\textit{given } g,\ k,\ \text{``etc.''}\ldots]{} m$$

Dependence on 'k':

- temperature
- non linearity
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- stopping position of damped oscillation;
- variability of all quantities of influence (in the ISO-GUM sense);
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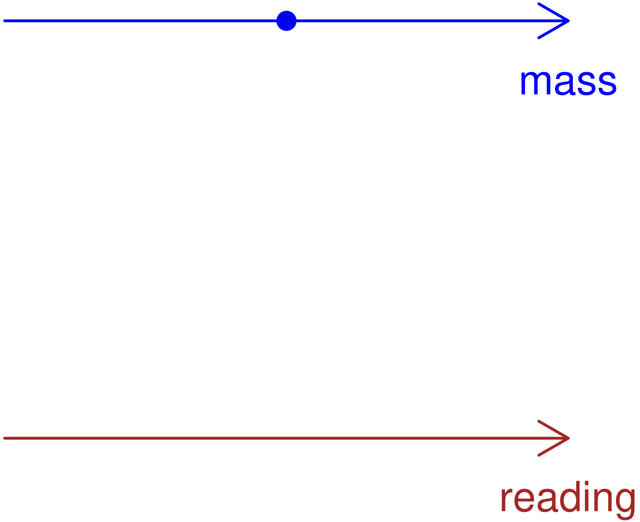
$$\Rightarrow \ m??$$

Mass $\longrightarrow$ Reading



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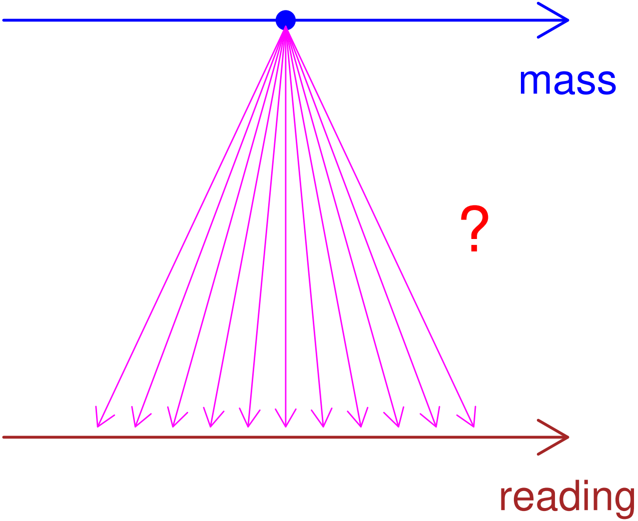

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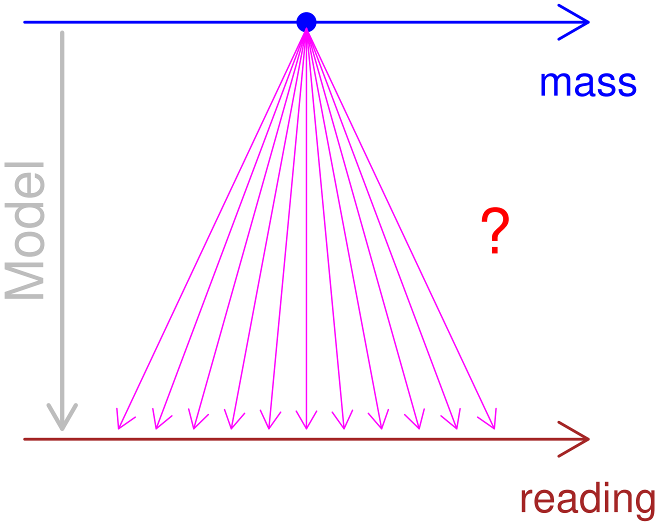

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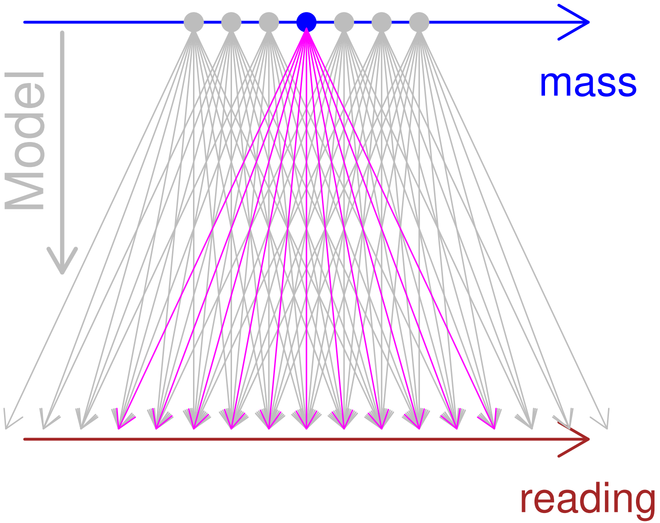

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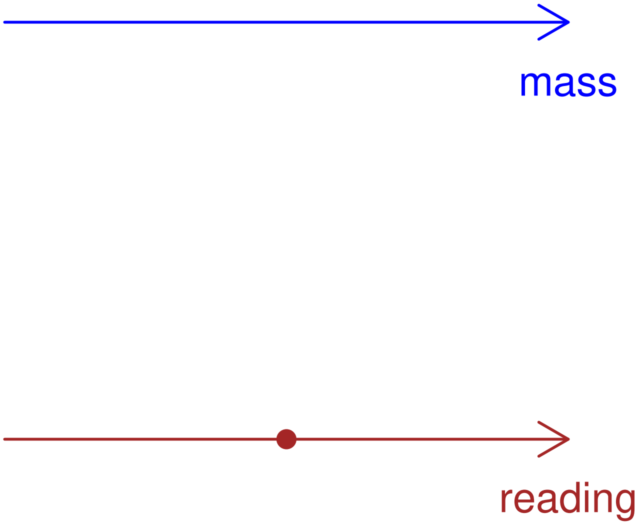

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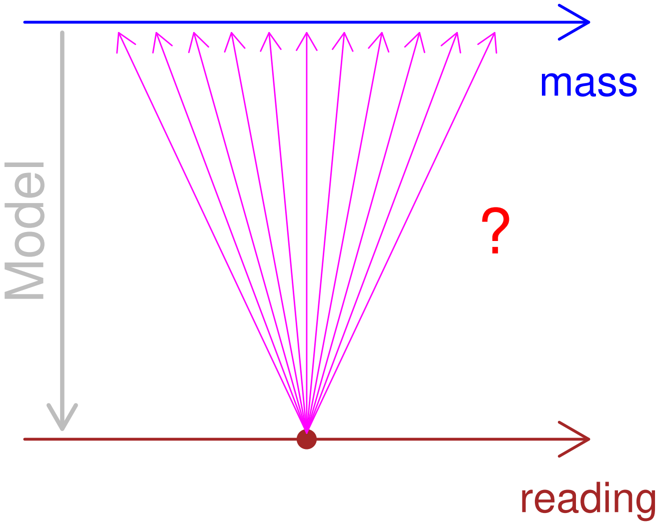

Sources of uncertainties (from ISO GUM)

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Note

- Sources not necessarily independent
- In particular, sources 1-9 may contribute to 10 (e.g. not-monitored electric fluctuations)

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Error and *uncertainty* are not synonyms!

Observation $\rightarrow$ value of a quantity

$$\text{scale reading} \quad \xrightarrow[\text{given } g,\ k,\ \text{“etc.”}\ldots]{} \quad m$$

Observations $\rightarrow$ hypotheses

This problem occurs not only "determining" *the* value of a physical quantity.

Observations → hypotheses

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- Experimental observation ('data') → responsible cause.

Observations $\rightarrow$ hypotheses

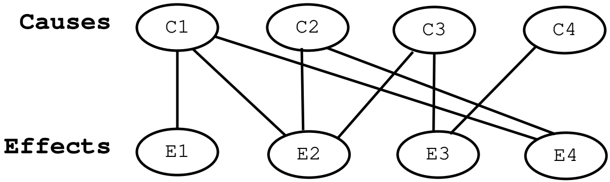
This problem occurs not only “determining” *the* value of a physical quantity.

- Experimental observation (‘data’) $\rightarrow$ responsible cause.

(But logically no substantial difference.)

Causes → effects

The same *apparent* cause might produce several,different effects



Given an observed effect, we are not sure about the exact cause that has produced it.

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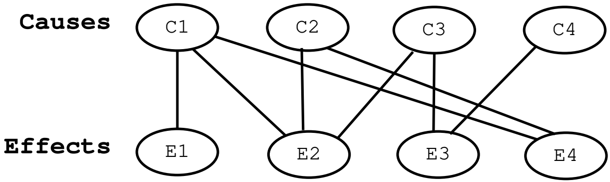

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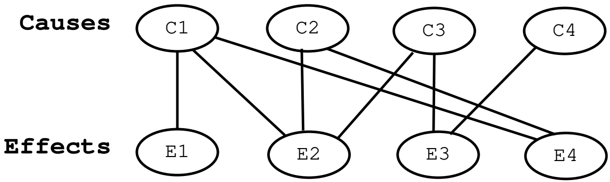

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$$\mathrm{E}_2 \Rightarrow \{C_1,\ C_2,\ C_3\}?$$

The “essential problem” of the Sciences

“Now, these problems are classified as *probability of causes*, and are most interesting of all for their scientific applications. I play at *écarté* with a gentleman whom I know to be perfectly honest. What is the chance that he turns up the king? It is $1/8$. This is a problem of the probability of effects.

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I play with a gentleman whom I do not know. He has dealt ten times, and he has turned the king up six times. What is the chance that he is a sharper? This is a problem in the probability of causes. It may be said that it is the essential problem of the experimental method.”

(H. Poincaré – *Science and Hypothesis*)

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Why we (or most of us) have not been taught how to tackle this kind of problems?

A minimalist though not trival problem of the kind

An example easy to understand:

- two causes;
- two effects;

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- medical diagnostics helps to clarify the issues:
  - easier to reach intuitive answers

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An example easy to understand:

- two causes;
- two effects;
- medical diagnostics helps to clarify the issues:
  - easier to reach intuitive answers
  - ... although if someone might have fallacious intuitions
    $\Rightarrow$ a formal guide helps us avoiding errors
    $\Rightarrow$ **logics of the uncertain** (theory of probabilities)

AIDS test

An Italian citizen is selected at random
to undergo an AIDS test.
→ Performance of clinical trial is not perfect, as customary:

$$
\begin{aligned}
P(\text{Pos}\,|\,\text{HIV}) &= 100\% \\
P(\text{Pos}\,|\,\overline{\text{HIV}}) &= 0.2\% \\
P(\text{Neg}\,|\,\overline{\text{HIV}}) &= 99.8\%
\end{aligned}
$$

H_1='HIV' (Infected) E_1 = Positive

H_2='$\overline{\text{HIV}}$' (Not infected) E_2 = Negative

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$$\begin{tikzcd} H_1 = \text{'HIV' (Infected)} \arrow[r] & E_1 = \text{Positive} \\ H_2 = \text{'}\overline{\text{HIV}}\text{' (Not infected)} \arrow[r] \arrow[ur] & E_2 = \text{Negative} \end{tikzcd}$$

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Result: $\Rightarrow$ <u>Positive</u>

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\begin{array}{lcl}
?\ H_1 = \text{'HIV' (Infected)} & \longleftarrow & E_1 = \text{Positive} \\
?\ H_2 = \text{'}\overline{\text{HIV}}\text{' (Not infected)} & \swarrow & E_2 = \text{Negative}
\end{array}
$$

Result: $\Rightarrow$ Positive

Infected or not infected?

AIDS test: how to interpret the result?

Being $P(\text{Pos}\,|\,\overline{\text{HIV}}) = 0.2\%$ and having observed 'Positive', can we say?

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(We will learn in the sequel how to evaluate it correctly)

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...from which bad decisions might follow!

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$\Rightarrow$ A sound formal guidance can rescue us

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Pay attention not to arbitrary revert conditional probabilities:

In general $P(A \mid B) \neq P(B \mid A)$

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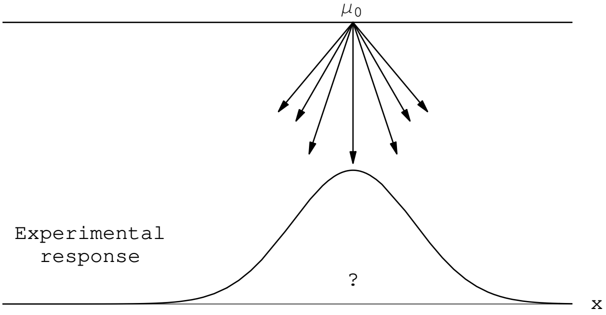
$\Rightarrow$ Misunderstanding **p-values** (a *related* logical mistake)

$\rightarrow$ Probability of causes

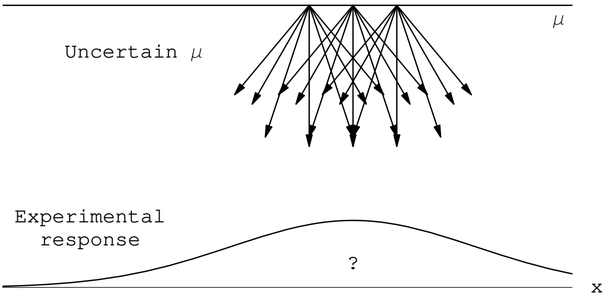
"the essential problem of the experimental method"

From 'true value' to observations



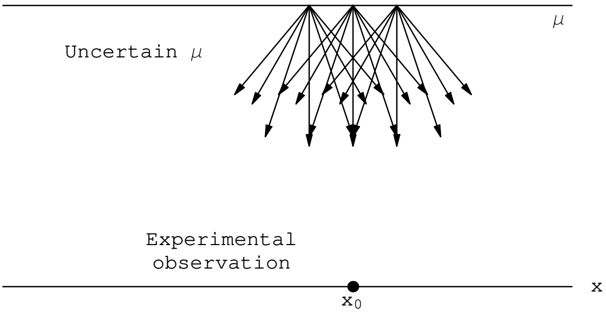
Given μ (exactly known) we are uncertain about x

From 'true value' to observations



Uncertainty about μ makes us more uncertain about x

. . . and back: Inferring a true value



The observed data is certain: $\rightarrow$ 'true value' uncertain.

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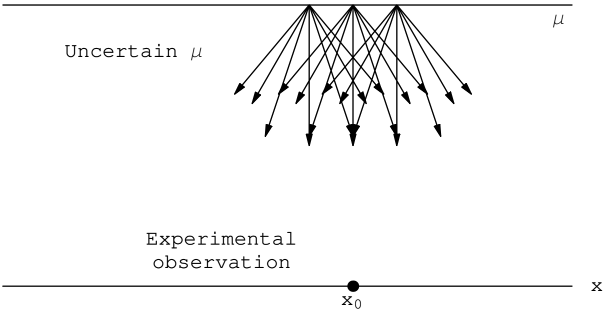

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"data uncertainty" ?

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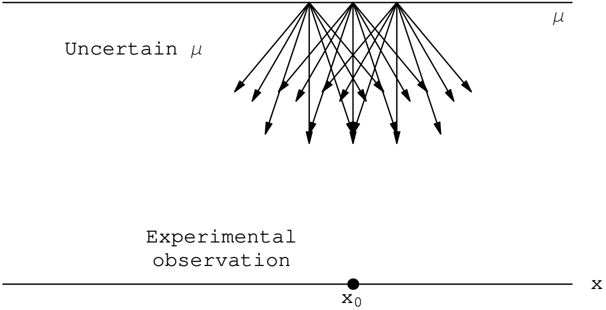

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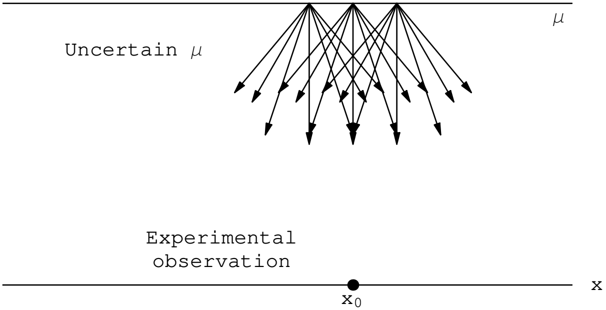

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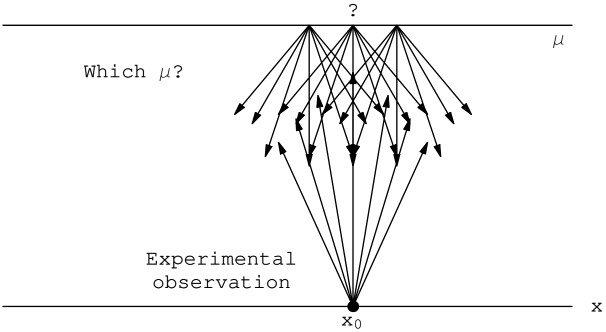
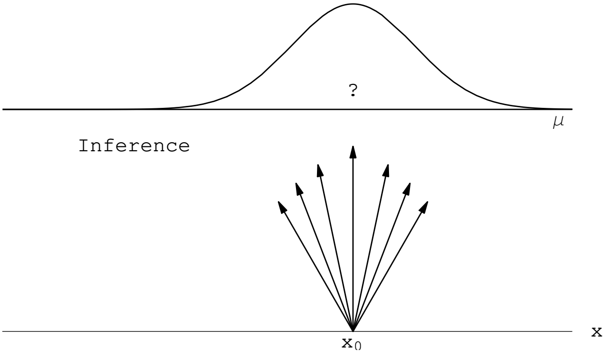
Even if the data were corrupted, the data were the corrupted data**!!** . . .

. . . and back: Inferring a true value



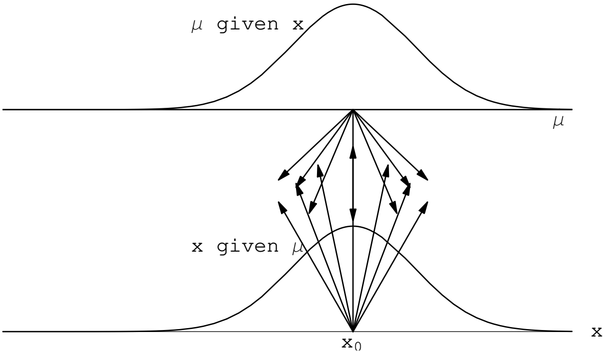
Where does the observed value of x comes from?

. . . and back: Inferring a true value



We are now uncertain about μ, given x.

. . . and back: Inferring a true value



Note the symmetry in reasoning.

Basic rules of probability

1. $0 \leq P(A \,|\, I) \leq 1$
2. $P(\Omega \,|\, I) = 1$
3. $P(A \cup B \,|\, I) = P(A \,|\, I) + P(B \,|\, I) \quad [\text{if } P(A \cap B \,|\, I) = \emptyset]$
4. $P(A \cap B \,|\, I) = P(A \,|\, B, I) \cdot P(B \,|\, I) = P(B \,|\, A, I) \cdot P(A \,|\, I)$

Remember that probability is always conditional probability!

I is the background condition (related to information 'I_s')

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⇒ easily extended to uncertain numbers ('*random variables*')

Subjective nature of probability

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Probability depends on the status of information of the *subject* who evaluates it.

Probability is always conditional probability

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$$P(E) \quad \longrightarrow \quad P(E \,|\, I_s(t))$$

where $I_s(t)$ is the information available to *subject* s at time t.

Mathematics of beliefs

An even better news:

The fourth basic rule
can be fully exploited!

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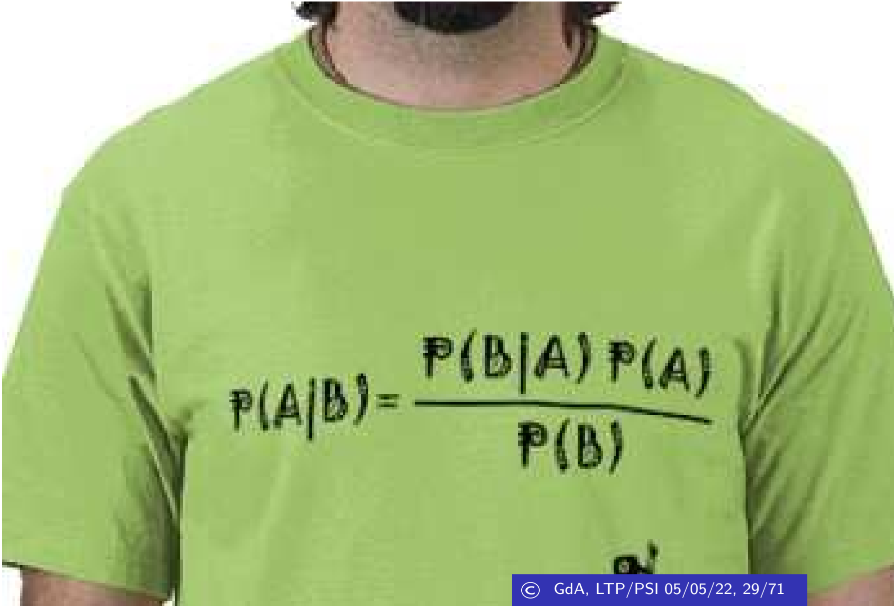
(Liberated by a curious ideology that forbids its use)

A simple, powerful formula



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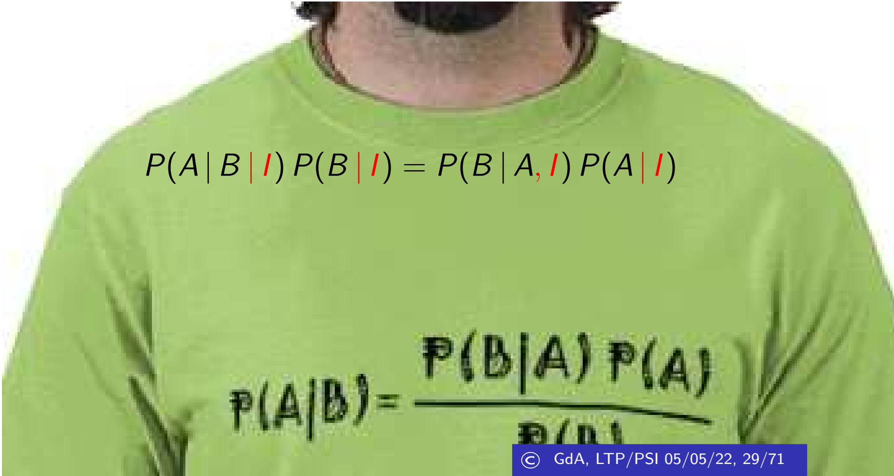

$$P(A \mid B \mid I)\, P(B \mid I) = P(B \mid A, I)\, P(A \mid I)$$

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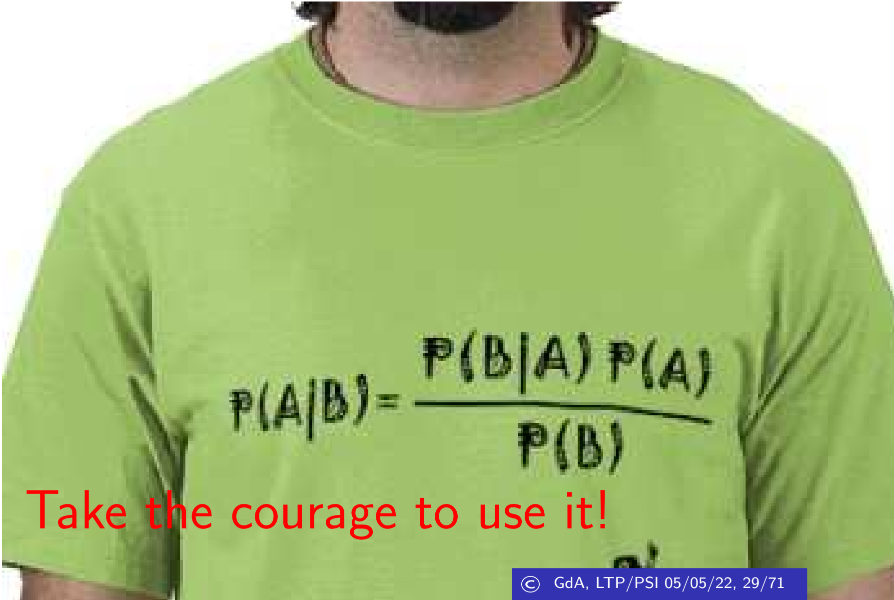
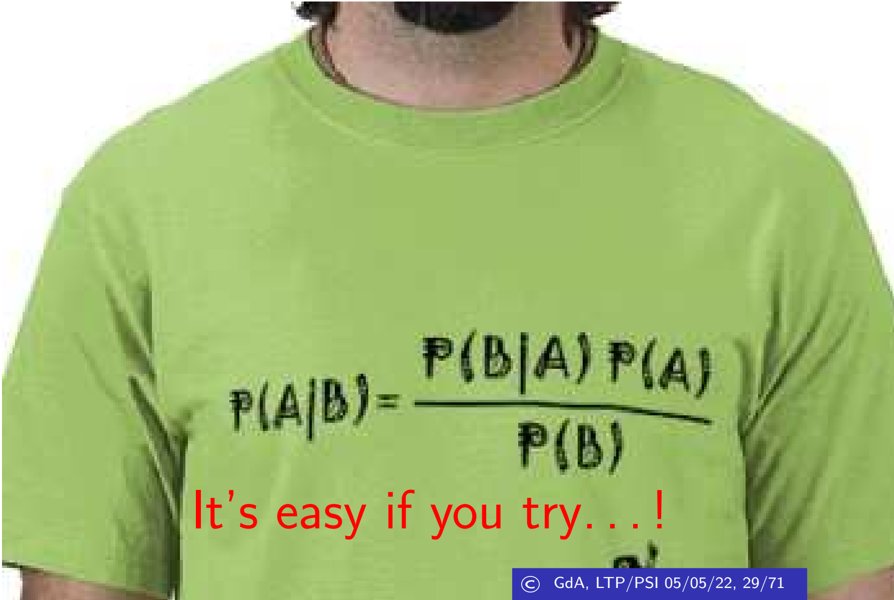


Take the courage to use it!

A simple, powerful formula



It's easy if you try. . . !

A simple, powerful formula

[Bayes Theorem]

Laplace's "Bayes Theorem"

"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}.

$$P(C_i \mid E) \propto P(E \mid C_i)$$

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"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}. The probability of the existence of any one of these causes {given the event} is thus a fraction whose numerator is the probability of the event given the cause, and whose denominator is the sum of similar probabilities, summed over all causes.

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$$P(C_i \,|\, E) = \frac{P(E \,|\, C_i)\, P(C_i)}{P(E)}$$

(*Philosophical Essai on Probabilities*)

[In general $P(E) = \sum_j P(E \,|\, C_j)\, P(C_j)$ (*weighted average*, with weigths being the probabilities of the conditions) if C_j form a *complete class of hypotheses*]

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"This is the fundamental principle $^{(*)}$ **of that branch of the analysis of chance that consists of reasoning a posteriori from events to causes"**

(*) In his "Philosophical essay" Laplace calls 'principles' the 'fundamental rules'.

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Most convenient way to remember Bayes theorem

Laplace's teaching

$$\frac{P(H_0 \,|\, \text{data})}{P(H_1 \,|\, \text{data})} = \frac{P(\text{data} \,|\, H_0)}{P(\text{data} \,|\, H_1)} \times \frac{P(H_0)}{P(H_1)}$$

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- If $P(\text{data} \,|\, H_i) = 0$, it follows $P(H_i \,|\, \text{data}) = 0$:
  $\Rightarrow$ falsification (the 'serious' one) is a corollary of the theorem, rather than a principle.

Laplace's teaching

$$\frac{P(H_0 \,|\, \text{data})}{P(H_1 \,|\, \text{data})} \;=\; \frac{P(\text{data} \,|\, H_0)}{P(\text{data} \,|\, H_1)} \times \frac{P(H_0)}{P(H_1)}$$

- We should possibly use the data, rather then the test variables 'θ' (χ^2 etc);

  [although in some case 'sufficient summaries' do exist]
- At least two hypotheses are needed!
- ...and also how they appear belivable *a priori*!
- If $P(\text{data} \,|\, H_i) = 0$, it follows $P(H_i \,|\, \text{data}) = 0$:

  $\Rightarrow$ falsification (the 'serious' one) is a corollary of the theorem, rather than a principle.
- There is no conceptual problem with the fact that $P(\text{data} \,|\, H_1) \rightarrow 0$ (e.g. 10^{-37}), provided the ratio $P(\text{data} \,|\, H_0)/P(\text{data} \,|\, H_1)$ is not undefined.

Bayes factor ('likelihood ratio')

$$\frac{P(H_0 \mid \text{data})}{P(H_1 \mid \text{data})} = \frac{P(\text{data} \mid H_0)}{P(\text{data} \mid H_1)} \times \frac{P(H_0)}{P(H_1)}$$

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$$\text{Prob. ratio}\big|_{posterior} = \text{Bayes factor} \times \text{Prob. ratio}\big|_{prior}$$

(*prior*/*posterior* w.r.t. data)

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$$\text{Prob. ratio}|_{posterior} \;=\; \text{Bayes factor} \times \text{Prob. ratio}|_{prior}$$

(*prior*/*posterior* w.r.t. data)

If H_0 and H_1 are 'complementary', that is $H_1 = \overline{H}_0$, then

$$\textbf{posterior odds} \;=\; \textbf{Bayes factor} \times \textbf{prior odds}$$

Telling it with Gauss' words

A quote from the Princeps Mathematicorum
(Prince of Mathematicians) is a must.

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$$P(C_i \,|\, \text{data}) = \frac{P(\text{data} \,|\, C_i)}{P(\text{data})}\, P_0(C_i)$$

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"post illa observationes" *"ante illa observationes"*

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Arguments used to derive Gaussian distribution

- $f(\mu\,|\,\{x\}) \propto f(\{x\}\,|\,\mu)\cdot f_0(\mu)$
- $f_0(\mu)$ 'flat' (all values a priory equally possible)
- posterior maximized at $\mu = \overline{x}$

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Note: indeed Gauss had also invented the "Bayes Factor"!
(GdA, arXiv:2003.10878 [math.HO])

Probabilistic inference/prediction

applied to the ‘binomial’ case

Namely the original problem tackled by Laplace and Bayes

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Namely the original problem tackled by Laplace and Bayes,
but in modern notation and making use of a graphical model:

Probabilistic inference/prediction

applied to the 'binomial' case

Namely the original problem tackled by Laplace and Bayes,
but in modern notation and making use of a graphical model:

1. draw the graphical model;
2. write down the joint pdf of all variables entering the game;
3. use Bayes theorem in order to condition on what is known/assumed;
4. *marginalize* over all variables on which we are not interesting;
5. do *somehow* the math.

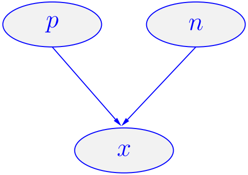
n independent *Bernoulli processes*

General case

n independent *Bernoulli processes*

General case

Model



n independent *Bernoulli processes*

General case

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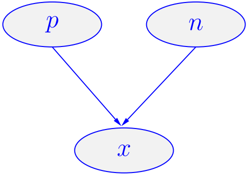

Joint pdf (omitting background condition I)
→ making use of the **chain rule**:

$$f(x, p, n) \;=\; f(x \,|\, p, n) \cdot f(p, n)$$

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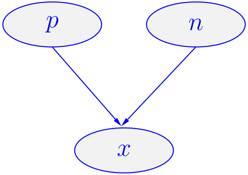

Joint pdf (omitting background condition I)
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\begin{aligned}
f(x, p, n) &= f(x \mid p, n) \cdot f(p, n) \\
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\end{aligned}
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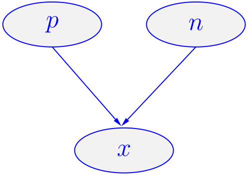

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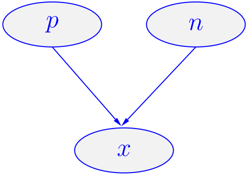

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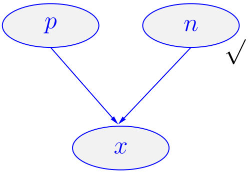
$$

(n and p are independent)

n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



n independent Bernoulli processes

Usual case $\rightarrow$ n fixed (for the moment)

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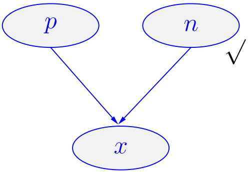
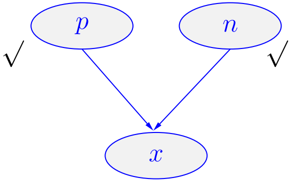


Joint pdf

$$f(x, p \mid n) \quad = \quad f(x \mid p, n) \cdot f(p)$$

n independent Bernoulli processes

Graphical models of the typical problems



$\rightarrow\ f(x \mid n, p)$

n independent Bernoulli processes

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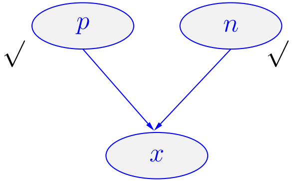

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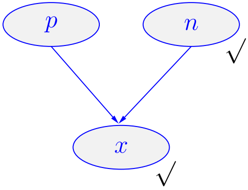

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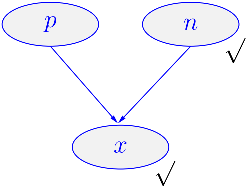
n independent Bernoulli processes

Inferring p



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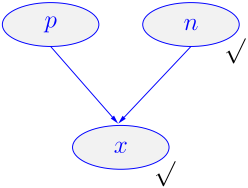

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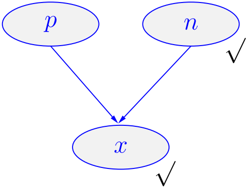

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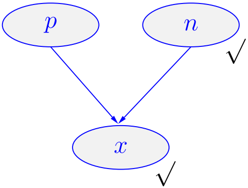

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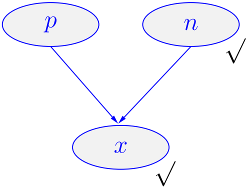

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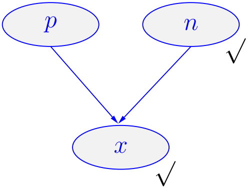

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(denominator just normalization!)

Inferring "Bernoulli's p"

We just need to make explicit $f(x \mid n, p)$:

$$f(x \mid n, p) \;=\; \binom{n}{x}\, p^x\, (1-p)^{n-x}$$

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(The binomial coefficient is irrelevant, not depending on p)

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For teaching purposes we start from a **uniform prior**, i.e. $\boldsymbol{f_\circ(p)} = 1$:

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- The integral at the denominator is the special function "β" (also defined for real values of x and n).
- In our case these two numbers are integer and the integral becomes equal to

$$\frac{x!\,(n-x)!}{(n+1)!}$$

Inferring "Bernoulli's p"

Solution for uniform prior

$$f(p \,|\, x, n) \;=\; \frac{(n+1)!}{x!\,(n-x)!}\, p^x\,(1-p)^{n-x}$$

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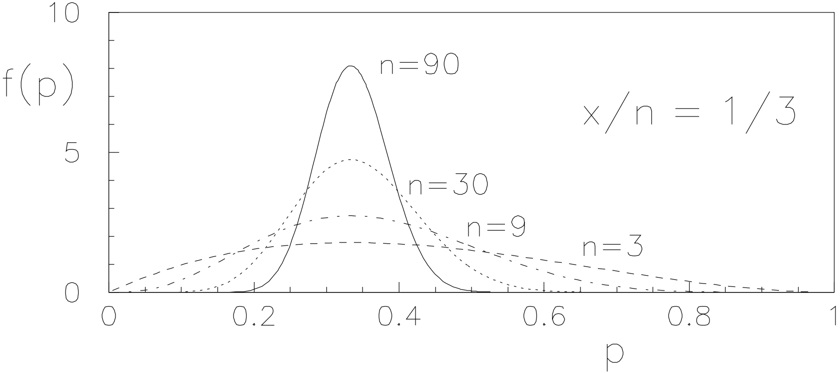

Inferring "Bernoulli's p"

Summaries of the **posterior distribution**

$$p_m = \mathrm{mode}(p) \quad = \quad \frac{x}{n}$$

Inferring "Bernoulli's p"

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$$\begin{aligned}
\mathrm{Var}(p) &= \frac{(x+1)(n-x+1)}{(n+3)(n+2)^2} \\
&= \frac{x+1}{n+2}\left(\frac{n+2}{n+2} - \frac{x+1}{n+2}\right)\frac{1}{n+3} \\
&= \mathrm{E}(p)\,(1-\mathrm{E}(p))\,\frac{1}{n+3}
\end{aligned}$$

Inferring the "Bernoulli's p"

About the meaning of E(p)

- We have used the "first"$^{(*)}$ n trials to learn about "p".

  [$^{(*)}$ "First" does not imply time order, but just order in usage.]

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Inferring the "Bernoulli's p"

About the meaning of $\mathrm{E}(p)$

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- But since we are uncertain about it, we have to take into account all possible values, weighing them with our degree of belief.

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Inferring the "Bernoulli's p"

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E(p) (and not the mode!) is the probability of every 'future' event which is believed to have the **same** $\boldsymbol{p}$ of the 'previous' ones.

Related recent applications

(with many details, including general introduction to the relevant ideas and methods, and program samples)

- GdA and A. Esposito, Checking individuals and sampling populations with imperfect tests, (arXiv:2009.04843 [q-bio.PE])
- GdA and A. Esposito, What is the probability that a vaccinated person is shielded from Covid-19? A Bayesian MCMC based reanalysis of published data with emphasis on what should be reported as 'efficacy', (arXiv:2102.11022 [stat.AP])

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  (In particular, the Gaussian is self-conjugate,
  which is not so great...)

Predictive distribution

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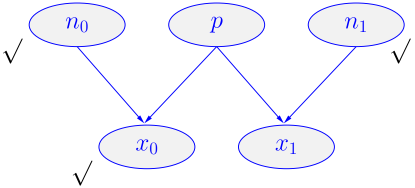

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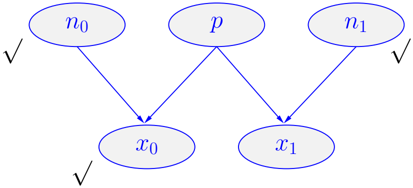

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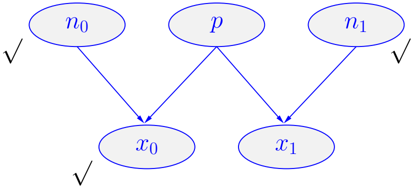

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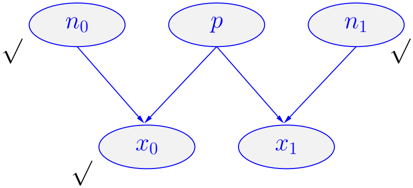

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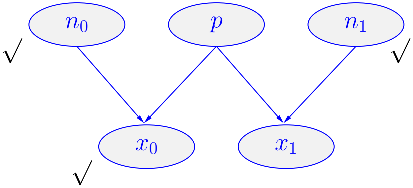

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- More precisely,

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- $X_1 \to f_1$ (Predicting a **future frequency from a past frequency**)

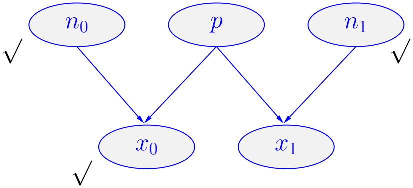
Predictive distribution

Some examples

$f(x_1 \mid n_0, x_0, n_1 = 10)$ in %

X_1	$\frac{X_1}{n_1}$	$\begin{cases} x_0 = 1 \\ n_0 = 2 \end{cases}$	$\begin{cases} x_0 = 10 \\ n_0 = 20 \end{cases}$	$\begin{cases} x_0 = 100 \\ n_0 = 200 \end{cases}$	$\begin{cases} x_0 = 1000 \\ n_0 = 2000 \end{cases}$
0	0	3.85	0.42	0.12	0.10
1	0.1	6.99	2.29	1.11	0.99
2	0.2	9.44	6.51	4.67	4.42
3	0.3	11.19	12.54	11.88	11.74
4	0.4	12.24	18.07	20.21	20.48
5	0.5	12.59	20.33	24.02	24.55
6	0.6	12.24	18.07	20.21	20.48
7	0.7	11.19	12.54	11.88	11.74
8	0.8	9.44	6.51	4.67	4.42
9	0.9	6.99	2.29	1.11	0.99
10	1	3.84	0.42	0.12	0.10
$\mathrm{E}(X_1)$		5	5	5	5
$\sigma[X_1]$		2.64	1.87	1.62	1.58

Joint inference and prediction



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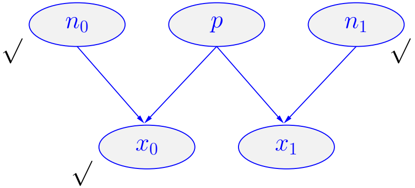

In reality the general solution starts from

$$f(n_0, p, n_1, x_0, x_1)$$

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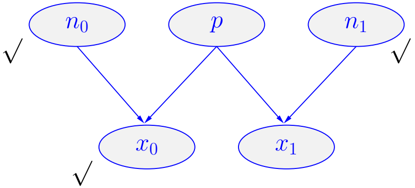

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conditioning on what is 'known' (or 'assumed'):

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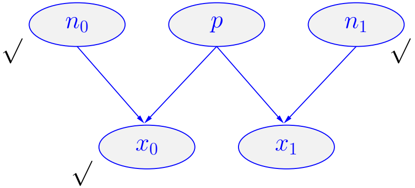

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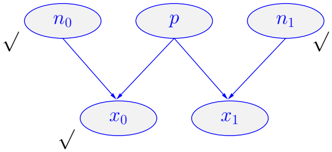
⇒ The denominator is just a constant.

⇒ **Very important observation** in order to solve the problem numerically or by Monte Carlo methods! (And remember that the numerator can be obtained using the chain rule)

Graphical models



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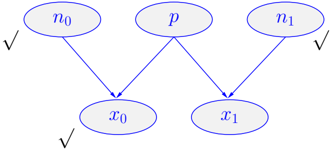

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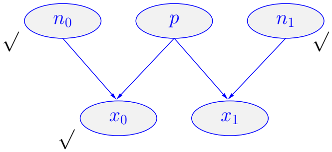

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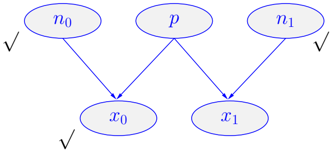

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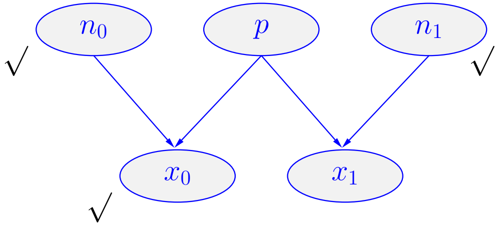
Software to analyse it:

- instructions which remind the description of the model by a suitable chain rule;
- computation performed by Markov Chain Monte Carlo.

Joint inference and prediction in JAGS

JAGS: Just Another Gibbs Sampler

(The Gibbs Sampler is an MCMC algorithm,
but the software also uses Metropolis in *though cases*)



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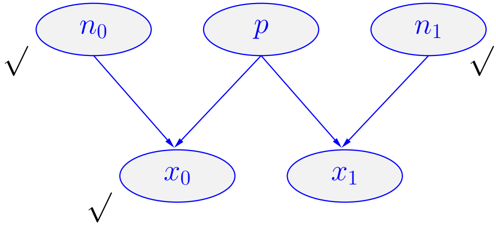
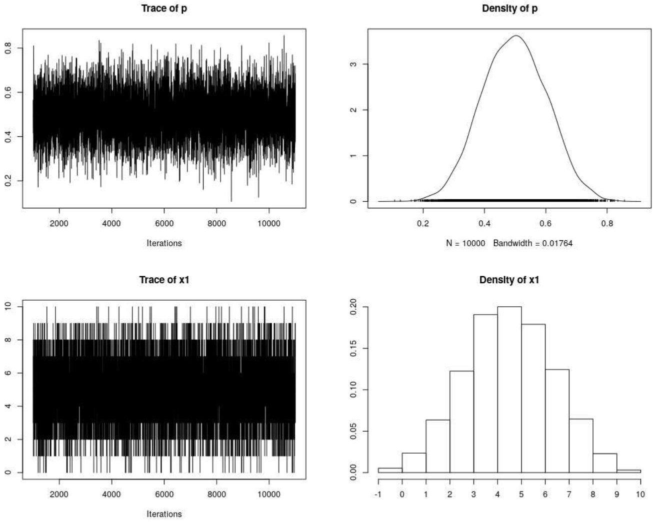


Model:

```
model{
    x0 ~ dbin(p, n0)
    x1 ~ dbin(p, n1)
     p ~ dbeta(1, 1)    # flat prior in termes of a Beta
}
```

Use of JAGS from R via rjags

$$(n0 = 20,\ x0 = 10,\ n1 = 10)$$



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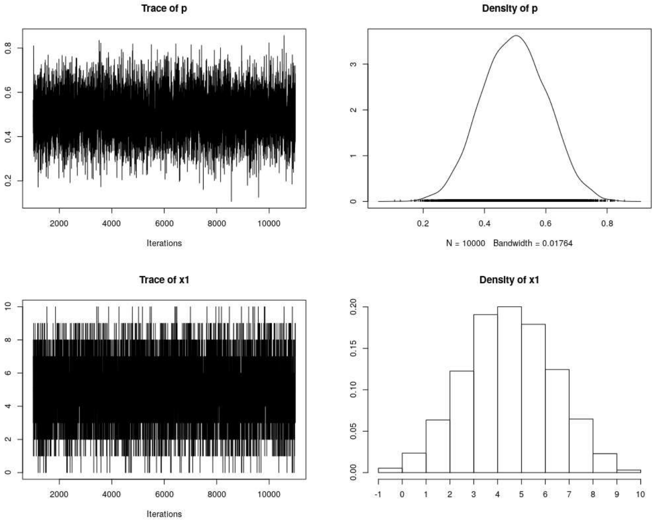
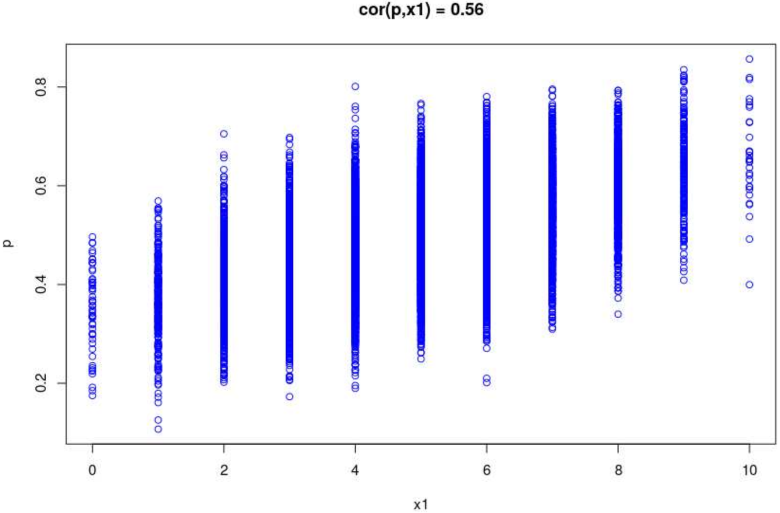


$p = 0.498 \pm 0.105$; $x_1 = 4.98 \pm 1.86$ (10000 samples).

Use of JAGS from R via rjags

Scatter plot of sampled $f(p, x_1 \mid n_0, x_0, n_1)$



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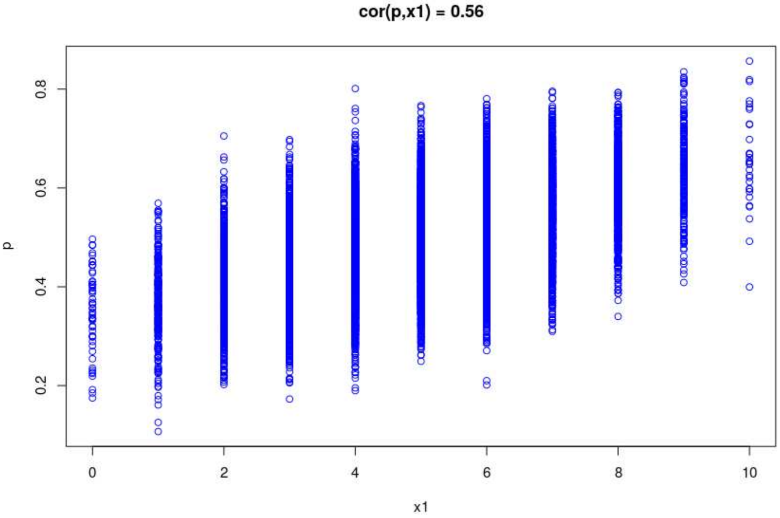

BAT - the Bayesian Analysis Toolkit

A package more suited for Physics analysis (expecially HEP)

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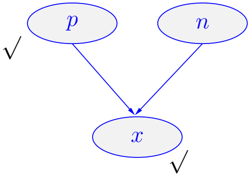
Presently rewritten in Julia: https://github.com/bat/BAT.jl

n independent Bernoulli processes

Inferring n



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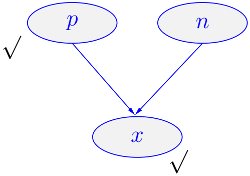

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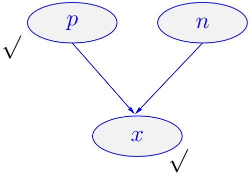

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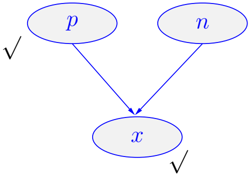

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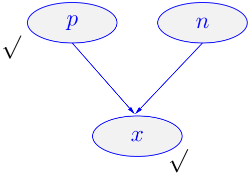

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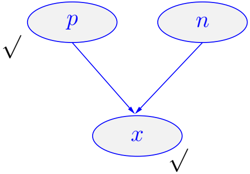
Extending the model

Our problem (but in Physics it is often not so simple)



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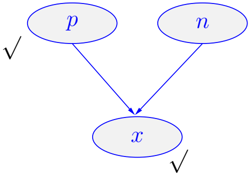

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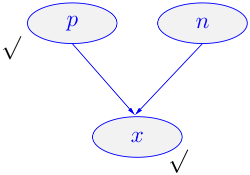

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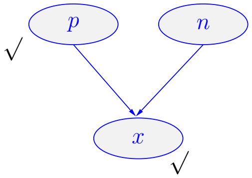

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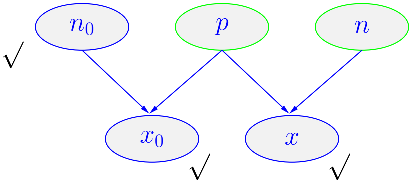

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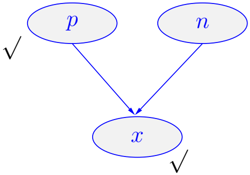

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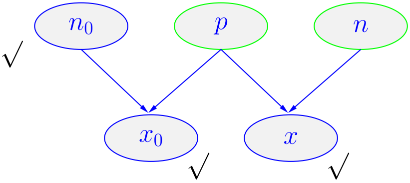

But what is n?

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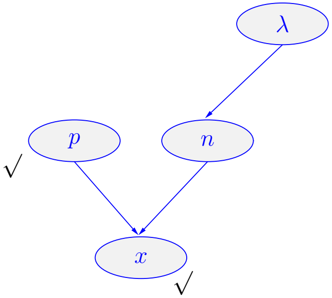

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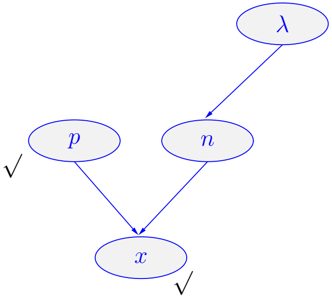

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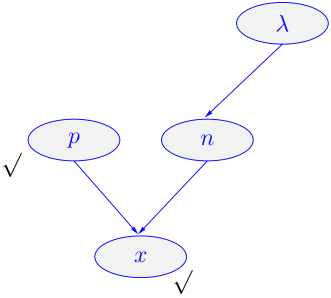
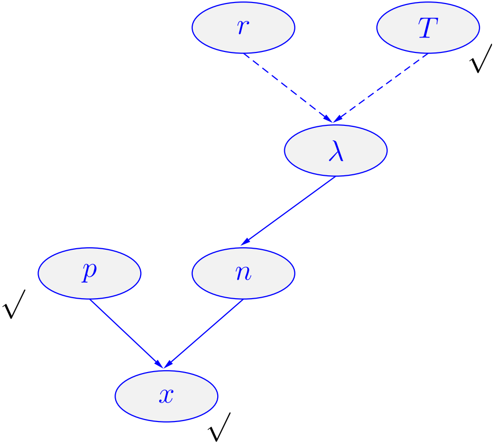


But λ is not really physical $\longrightarrow \lambda = r\,T$

(r: intensity of the Poisson process).

Extending the model

$\boldsymbol{\lambda} = r \cdot T$:



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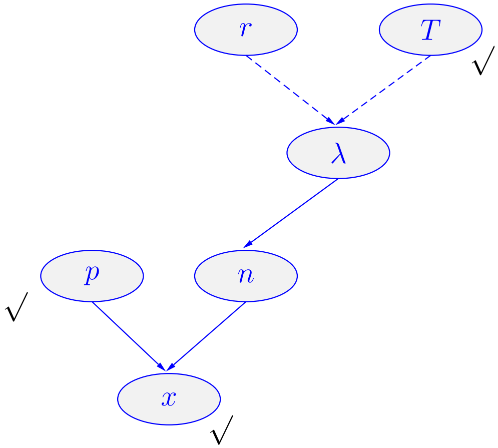

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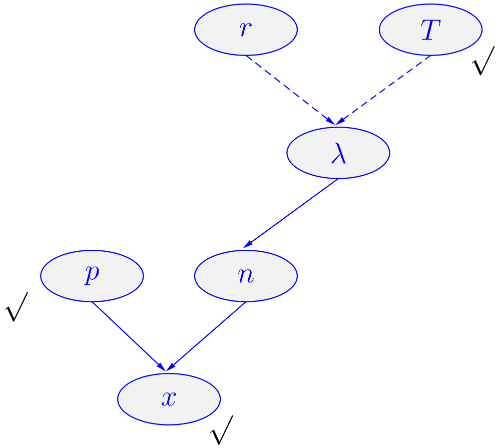

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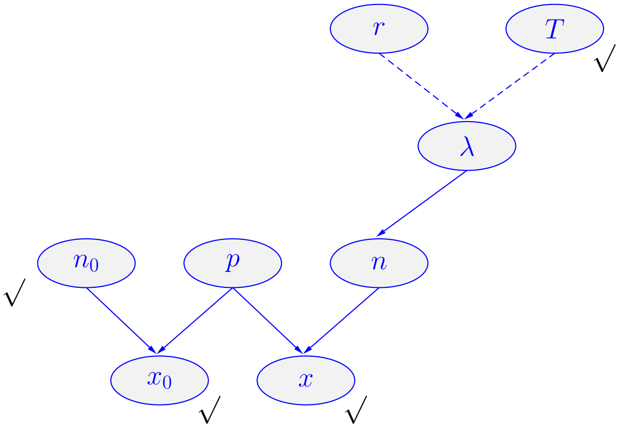
In JAGS, e.g., `lambda <- r * T;`

Extending the model

Remembering that p was got from a measurement:



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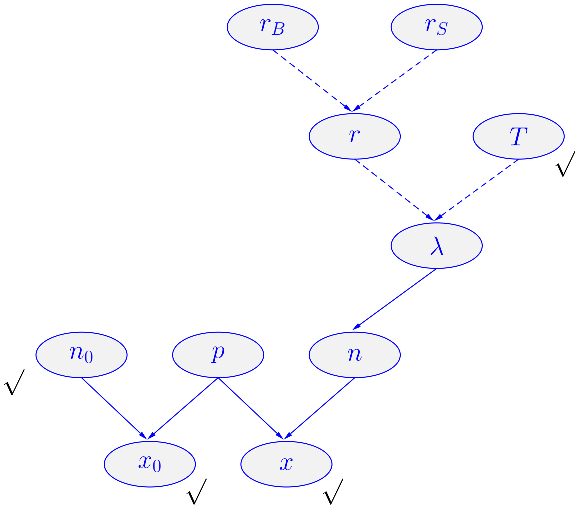

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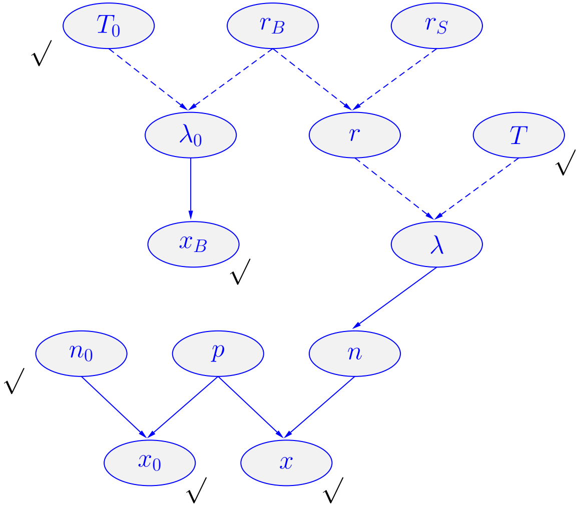

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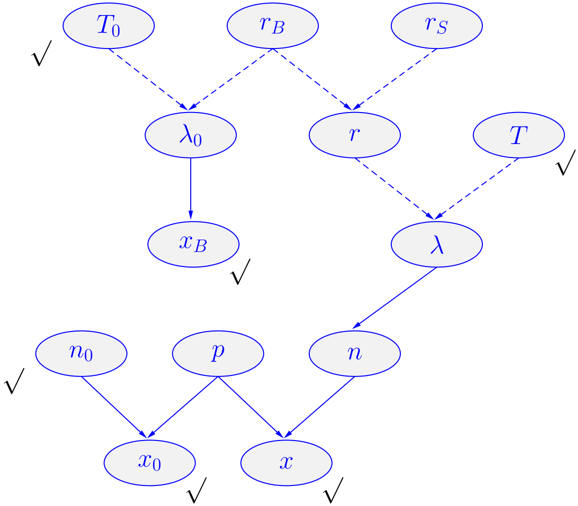

(T_0 and T assumed to be measured with sufficient accuracy)

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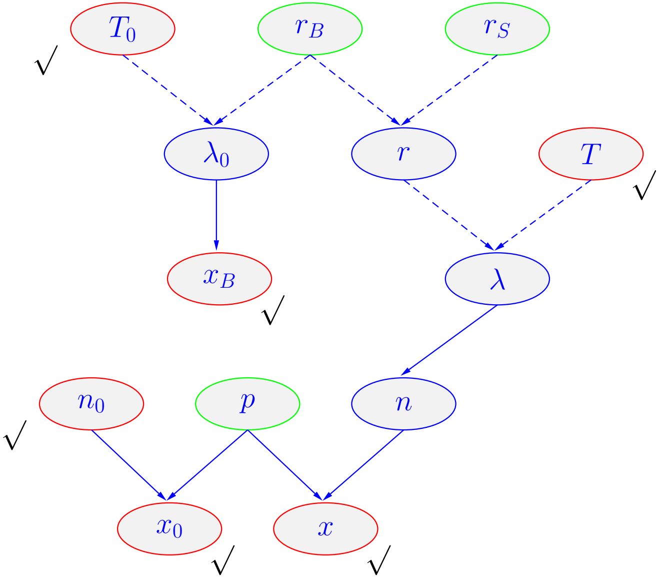

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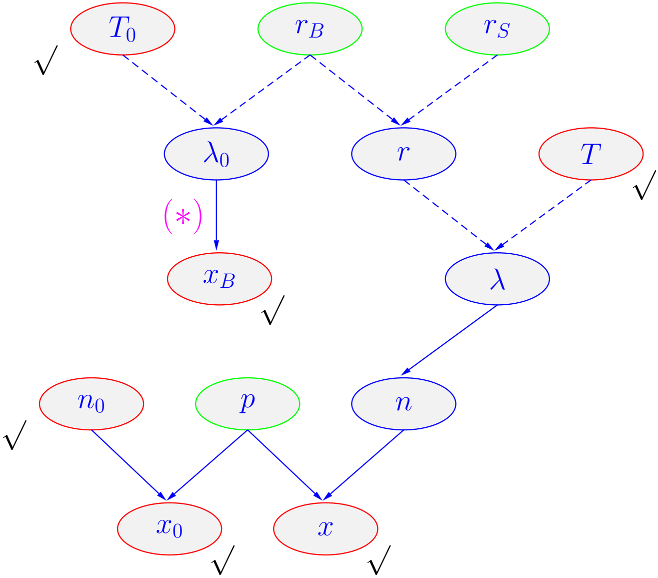
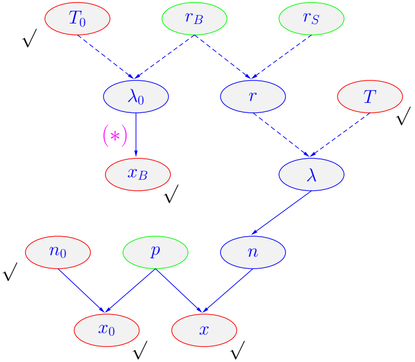


(∗) Assuming unity efficiency

Extending the model



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- writing down the joint pdf of all variables;
- (re-)conditioning on the assumed/observed quantities;
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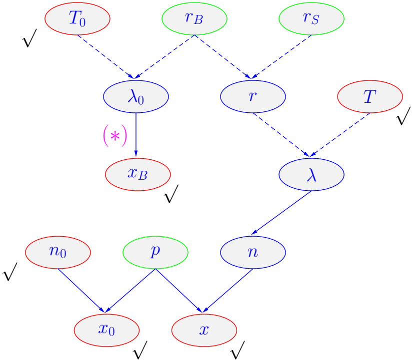

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Or, more easily, use software **grounded on probability theory**, like BUGS, JAGS, BAT, etc.

Other interesting models

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More detailed applications (including scripts) in

- `https://www.roma1.infn.it/~dagos/prob+stat.html`

[For BAT and BAT.jl see their web pages]

Summing up

- The probabilistic framework basically set up by Laplace$^{(*)}$ in his monumental work is healthy and grows up well.

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- The probabilistic framework basically set up by Laplace$^{(*)}$ in his monumental work is healthy and grows up well.
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  - you get a superior awareness of how the variables are related;
  - you are very close to the solution, although the analitic one is often out of reach (MCMC's rescue us!)

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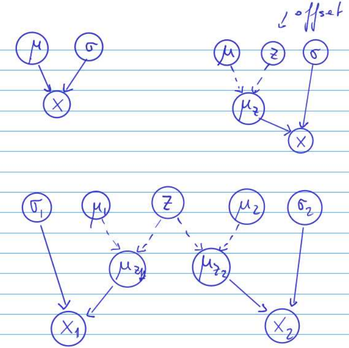
Handling systematics in the probabilistic approach

(Answering to a question: diagrams show the case of uncertain offset systematics, best known to be, after proper calibration, $z = 0 \pm \sigma_z$)



1) Joint pdf : $f(\cdots\cdots)$

2) $\Rightarrow$ $f(\mu_1, \mu_2 \,|\, \ldots\ldots)$

$\Rightarrow$ details in Bayesian Reasoning (and CERN Report)

References

- As starting points, particularly recommended are the last papers, in which graphical models are systematically exploited:
  - arXiv:2001.03466 [physics.data-an]
  - arXiv:2009.04843 [q-bio.PE]
  - arXiv:2012.04455 [stat.ME]
  - arXiv:2102.11022 [stat.AP]
- Much more can be found in
  - `https://www.roma1.infn.it/~dagos/prob+stat.html`
  - `https://www.roma1.infn.it/~dagos/dott-prob_31/`